Change of the Adiabatic Invariant due to Separatrix Crossing

J. L. Tennyson[(a)]
Institute for Fusion Studies, University of Texas, Austin, Texas 78712

John R. Cary
Department of Astrophysical, Planetary, and Atmospheric Sciences, University of Colorado, Boulder, Colorado 80309

and

D. F. Escande
Laboratoire de Physique des Milieux Ionisés, Ecole Polytechnique, 91228 Palaiseau, France
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When a parameter in the Hamiltonian of a one-degree-of-freedom oscillator is slowly varied at rate ϵ, an adiabatic invariant exists which is conserved to all orders in ϵ, except on phase-space orbits which cross a separatrix. In the present work, the change in the adiabatic invariant due to a separatrix crossing is given to order ϵ for a wide class of Hamiltonian systems. This result is applied to the special case of a charged particle moving under the influence of an electrostatic wave with slowly varying amplitude and frequency.

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The theory of adiabatic invariance in classical Hamiltonian systems breaks down when phase-space orbits cross separatrices of the phase flow. In this work, the behavior of separatrix-crossing orbits is described for a wide class of Hamiltonian systems. These systems are represented by Hamiltonians of the form $H(q,p,\lambda)$ whose phase-space contours include a generic separatrix consisting of two lobes connected by an x point (see Fig. 1). Examples are double-minimum potential wells and (less obviously) the simple pendulum. When the parameter λ is slowly varied at rate $\epsilon = d\lambda/dt$ between two values λ_i and λ_f, a near-invariant of the motion $J(q,p,\lambda;\epsilon)$, called an adiabatic invariant, exists in the form of an asymptotic power series in ϵ, the first term of which is the action $I(q,p,\lambda)$:

$$J(q,p,\lambda;\epsilon) \approx I(q,p,\lambda) + \epsilon J_1(q,p,\lambda) + \epsilon^2 J_2(q,p,\lambda) + \dots \quad (1)$$

The truncations of this series are ϵ-dependent phase functions which are conserved to the order of truncation, except when the orbit crosses a separatrix during the variation.[1,2] In this Letter, the change in the value of the adiabatic invariant (of all truncations of the series above first order) is given for separatrix-crossing orbits to order ϵ. A more detailed study[3] will show the error calculations and the statistical implications of these results.

This work was motivated by the need for a precise description of separatrix crossing in a number of diverse applications. Action-variation calculations have been applied to studies of energy and momentum balance for waves in collisionless plasmas,[4–6] while slow separatrix crossings have been discussed in studies of mirror-containment degeneration due to low-frequency fluctuations[7] and particle transport in strongly turbulent plasmas.[8] Slow separatrix crossings also occur in high-energy accelerators, where coasting particles are trapped and bunched by slowly ramped rf fields, and in colliding-beam storage rings, where betatron amplitudes can diffuse due to repeated slow crossings of beam-beam resonances.[9] Applications also exist for free-electron lasers, where electrons cross or are trapped by phase-space buckets, and in celestial mechanics, e.g., in the analysis of pairs of satellites with near-commensurable mean motions.[10] Many celestial-mechanics applications are discussed by Henrard.[11]

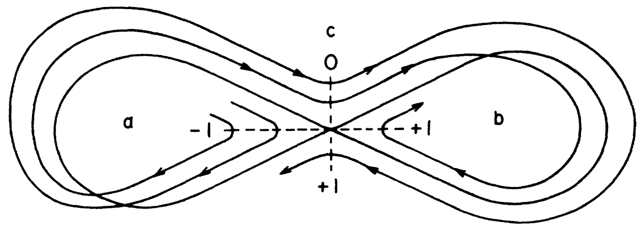

FIG. 1. Separatrix crossing orbits for $-Y_a' > Y_b' > 0$. Single- and double-crossing orbits are shown relative to a generic separatrix. The positions of the $n = -1, 0, +1$ vertices are indicated.

To avoid confusion in the application of these results, the following conventions are adopted. Suitable coordinate transformations are made such that the energy h (the value of the Hamiltonian function) is zero on the separatrix. The three regions separated by the separatrix (see Fig. 1) are labeled $\alpha = a$, b, and c, where c is the outside region. The action $I_\alpha(q,p,\lambda)$ at a particular phase point q,p in region α is defined to be the integral $\int p\,dq$, in the direction of the phase flow, around the closed contour of the Hamiltonian at λ which intersects q,p (note that the actions may be negative). The actions associated with the separatrix itself are denoted $Y_a(\lambda)$ and $Y_b(\lambda)$ for the two lobes, with $Y_c(\lambda)=Y_a(\lambda)+Y_b(\lambda)$. A near-separatrix orbit passes close to the x-point once each period (twice, when it is in region c). The points of closest approach to the x point are called "vertices." Each vertex of a particular orbit is labeled with an integer $n=\ldots,-2,-1,0,1,2,\ldots$, ordered as the vertices occur in time and such that the $n=0$ vertex is always the outside vertex closest to the point of crossing (although the vertices of a particular orbit are defined relative to a certain choice of metric, the results ΔJ obtained below do not depend on this choice). The energy and value of λ at the nth vertex are denoted h_n and λ_n. Lobe a is assumed to be the lobe whose encirclement by the orbit immediately precedes the $n=0$ vertex. The exponentiation rate of motion near the x point is denoted by ω, and each lobe is characterized by the derivative of the lobe area with respect to λ at λ_0,

$$Y'_\alpha=\partial Y_\alpha/\partial\lambda,\quad \alpha=a,b,c,$$

and the constant

$$h_\alpha=\lim_{h\to 0}\{h\exp[\omega T_\alpha(h)]\},\quad \alpha=a,b,$$

where $T_\alpha(h)$ is the period of orbit in lobe α with energy h at $\lambda=\lambda_0$. Actions are approximately conserved. Therefore, as the two lobes slowly change size, most orbits cross the separatrix separating the three regions a, b, and c. There are two general types of crossings: single crossings which cross to or from the external region, and double crossings, which cross from one lobe to another.

The calculation of ΔJ is performed as follows. The change in the first-order adiabatic invariant $J^1=I+\epsilon J_1$ is calculated for the N individual steps (orbit segments between consecutive vertices) on either side of the actual crossing. The $2N$ step changes are then summed to give the change in J^1 over the entire near-separatrix part of the orbit. It is then shown that as $N\to\infty$, this sum is equal (to relevant order) to the full change ΔJ^1 that occurs on a crossing trajectory between two fixed values of λ.

The calculation of the step change in J^1 is based on the approximation that for near-separatrix orbits in region α, the nth step change Δh in energy and the nth step change in $\Delta\lambda$ in λ are given by

$$\Delta h_\alpha=-\epsilon Y'_\alpha,\quad \alpha=a,b,c, \tag{2}$$

$$\Delta\lambda_\alpha=(\epsilon/2\omega)\{\ln(h_\alpha/h_n)+\ln[(h_\alpha/(h_n+\Delta h_\alpha)]\},\quad \alpha=a,b, \tag{3}$$

even when the orbit crosses the separatrix during the step. It follows from Eq. (2) that the $2N$ steps span an energy interval of order ϵ, and that within this interval, h may be assumed to be small. This leads to a small-energy approximation to J^1 at the vertices which depends on h, λ, and four separatrix constants $Y_\alpha(\lambda_0)$, $Y'_\alpha(\lambda_0)$, h_α, and g_α (defined below):

$$J^1_\alpha(h,\lambda,\epsilon)=I_\alpha(h,\lambda)+\epsilon[g_\alpha+f_\alpha(h)]+o(h^{3/2})+O(\epsilon h),\quad I_\alpha(h,\lambda)=Y_\alpha(\lambda)+(h/\omega)(1+\ln|h_\alpha/h|),\quad \alpha=a,b,$$

$$g_\alpha=\lim_{h\to 0}\left\{\frac{Y'_\alpha\ln|h_\alpha/h|}{2\omega}-\oint dq\frac{\partial P}{\partial h}\int_{q_0}^{q}dq'\frac{\partial P}{\partial\lambda}\right\},\quad \alpha=a,b,\quad f_c(h)=\frac{Y'_a\ln|h_b/h|-Y'_b\ln|h_a/h|}{2\omega}, \tag{4}$$

with $g_c=g_a+g_b$, $I_c=I_a+I_b$, and $f_a=f_b=0$. The function $P(q,h,\lambda)$ is found by inversion of the Hamiltonian equation $H(p,q,\lambda)=h$, i.e., solving for p in terms of (q,h,λ). The integrals are taken around the contour of the Hamiltonian defined by h at $\lambda=\lambda_0$, in the direction of motion, with q_0 the value of q at the vertex of the contour. The constant g_α is zero if lobe α is symmetric (up-down, in Fig. 1). The function $f_c(h)$ is zero if the separatrix is symmetric with respect to reflection about the x point. With use of Eqs. (2), (3), and (4), the calculation of the step change in J^1 is straightforward. The sum of the $2N$ step changes gives an approximation to the net change in J^1 over the full orbit (the sum converges as $N\to\infty$). For orbits that cross from region α to region β, the net change in J^1 is given by

$$\Delta J^1(h_0,\lambda_0)=S_\beta(h_0)+J^1_\beta(h_0,\lambda_0)+S_\alpha(h_0)-J^1_\alpha(h_0,\lambda_0)+O(\epsilon^{6/5}),$$

$$S_\alpha(h_0)=\frac{\epsilon Y'_\alpha}{\omega}\left\{(m_\alpha-\tfrac{1}{2})\ln|m_\alpha|-m_\alpha+\ln\left(\frac{\Gamma(1-m_\alpha)}{2\pi^{1/2}}\right)\right\},\quad \alpha=a,b, \tag{5}$$

$$S_c(h_0)=\frac{\epsilon Y'_c}{\omega}\left\{-(2m_c+R_\gamma)\ln|m_c|+2m_c+\ln\left(\frac{\Gamma(m_c+1)\Gamma(m_c+R_\gamma)}{2\pi}\right)\right\},$$

where $m_\alpha = |h_0/\Delta h_\alpha|$, $R_\gamma = \Delta h_\gamma/\Delta h_c$, and $\Gamma(x)$ is the usual gamma function. The subscript γ is the label a or b of the lobe not entered (or left) by a single-crossing orbit. All of the explicit terms in Eq. (5), except the Y_α, are order ϵ or order $\epsilon \ln\epsilon$. The result (5) is independent of the choice of metric. Although the constants g_α and $Y_\alpha(\lambda_0)$ depend on the metric, the sum $Y_\alpha(\lambda_0) + \epsilon g_\alpha$ does not.

If Δh_a and Δh_b have different signs, $|h_0|$ varies from zero to the larger of $|\Delta h_a|$, $|\Delta h_b|$, with the smaller of them dividing this range into two parts. The part with the smaller values of $|h_0|$ gives double crossings while the part with the larger values gives single crossings. As a result of precrossing phase mixing, any smooth distribution of initial conditions defines an ensemble in which every value of h_0 (within the above range) is equally probable. Thus the probability of a double crossing is given by the smaller of $|\Delta h_a/\Delta h_b|$, $|\Delta h_b/\Delta h_a|$. This result was proved by Neishtadt[12] and Henrard.[11]

As an illustration, the above formula is applied to a well-known (but previously unsolved) problem. The system consisting of a charged particle moving under the influence of an electrostatic wave with slowly varying amplitude $(1+\dot{A}t)$ and frequency $\dot{\Omega} t$ is defined by the Hamiltonian

$$H(p,q,t) = \tfrac{1}{2}p^2 + (1+\dot{A}t)\cos(q - \tfrac{1}{2}\dot{\Omega} t^2).$$

A canonical transformation puts this into a form consistent with the above conventions (with positive energy lobes)

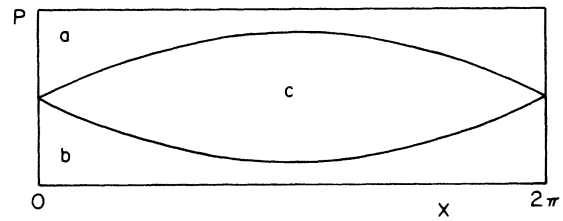

FIG. 2. The wave-particle separatrix and its correspondence to the generic separatrix.

$$H(P,Q,t) = \tfrac{1}{2}(P - \dot{\Omega} t)^2 - 2(1+\dot{A}t)\sin^2 \tfrac{1}{2}Q.$$

The areas inside and above the wave separatrix in Fig. 2 correspond to the regions c and a, respectively, in Fig. 1. The various parameters that determine the change in action (5) are

$$\epsilon Y_a' = (4\dot{A} + 2\pi\dot{\Omega}), \quad \epsilon Y_b' = (4\dot{A} - 2\pi\dot{\Omega}),$$

$$h_a = h_b = 32, \quad g_a = g_b = 0, \quad \omega = 1,$$

$$Y_a(\lambda_0) = Y_b(\lambda_0) = 8, \quad f_c(h_0) = 2\pi\dot{\Omega}\ln|h_0/32|.$$

In previous studies of this system, the definitions of the actions have differed slightly from the conventions chosen here. If we define the "wave standard" invariants $\mathscr{J}_a^1 = J_a^1$, $\mathscr{J}_b^1 = -J_b^1$, and $\mathscr{J}_c^1 = \frac{1}{2}J_c^1$, the changes in the wave-standard adiabatic invariant are immediately obtained from Eq. (5). For single and double crossings, respectively, these are

$$\Delta_s\mathscr{J}^1(h_0,\lambda_0) = \tfrac{1}{2}\{S_c(h_0) + J_c^1(h_0,\lambda_0)\} + S_a(h_0) - J_a^1(h_0,\lambda_0), \tag{6}$$

$$\Delta_d\mathscr{J}^1(h_0,\lambda_0) = -\{S_b(h_0) + J_b^1(h_0,\lambda_0)\} + S_a(h_0) - J_a^1(h_0,\lambda_0). \tag{7}$$

For the special case $\dot{\Omega} = 0$ where only single crossings are possible (particles become trapped in the wave), the symmetries reduce $\Delta_s\mathscr{J}^1$ to

$$\Delta_s\mathscr{J}^1(h_0,\lambda_0) = -4\dot{A}\ln[2\sin(\pi m_s)], \tag{8}$$

where $m_s = |h_0/-4\dot{A}|$. This was derived by Timofeev[13] in 1978. For the special case $\dot{A} = 0$ where only double crossings occur

$$\begin{gathered}\Delta_d\mathscr{J}^1(h_0) = 2S_a(h_0) - 16 - 2h_0(1+\ln|32/h_0|),\\ S_a(h_0) = 2\pi\dot{\Omega}\{(m_d - \tfrac{1}{2})\ln|m_d| - m_d + \ln[\Gamma(1-m_d)/(2\pi)^{1/2}]\},\end{gathered} \tag{9}$$

where $m_d = |h_0/2\pi\dot{\Omega}|$. This gives the change in the momentum $\Delta p = \Delta_d\mathscr{J}^1/2\pi$ of a particle caused by its being passed, in velocity space, by an accelerating wave. Note that for both special cases (8) and (9), $\Delta\mathscr{J}^1$ is equal to the change in the wave-standard action.

The error in the general expression (5) for ΔJ^1 will be small if the larger of $|\Delta h_a|$, $|\Delta h_b|$ is much less than the smaller of h_a, h_b, and if h_0 does not get too close to the extremes of its range [e.g., provided $|h_0| > h_\beta \exp(-\omega|Y_\alpha(\lambda_0)/\Delta h_\alpha|)$, for any combination of regions $\alpha, \beta = a,b$].

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[(a)]Present address: School of Mathematical Sciences, Queen Mary College, Mile End Rd., London E1 4NS, United Kingdom.

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