Shape Selection of Saffman-Taylor Fingers

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Among all problems of pattern selection the one posed by the Saffman-Taylor finger is yet unsolved, although numerics and experiments indicate the very simple property that the relative width of the finger tends to $\frac{1}{2}$ in the low–surface-tension limit. We explain this by performing an expansion beyond all orders leading to the formulation of a nonlinear eigenvalue problem with a discrete set of solutions. In particular we predict that the relative width of the finger tends to $\frac{1}{2}$ as the surface tension to the power $\frac{2}{3}$.

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Pattern selection in nonequilibrium phenomena, such as dendritic solidification or multiphase fluid flow, is a subject of much recent interest. The determination of the width of Saffman-Taylor fingers is a primary example in that field. Saffman and Taylor[1] have studied the displacement of a viscous fluid by a less viscous one in a Hele-Shaw cell (a rectangular cavity between narrowly spaced glass plates) in an attempt to model displacement of oil by water in porous media. A single finger of the less-viscous fluid is eventually formed and propagates at constant velocity keeping a steady shape. In the absence of interfacial tension between the two fluids, Saffman and Taylor have computed explicitly the interface shape. They have obtained a continuous family of fingers parametrized by the relative width λ between the finger and the channel. What determined the actual shape of the finger was not understood. Then, McLean and Saffman[2] derived the equations for the finger profile in the presence of interfacial tension. Their numerical solutions[3] give a discrete set of possible relative finger widths which surprisingly all decrease to $\frac{1}{2}$ when the dimensionless surface tension goes to zero. This was quite mysterious because attempts treating interfacial tension as a conventional perturbation were unable to detect any sign of selection. Very recently, in the problem of velocity selection in dendritic growth, some evidence has been produced numerically,[4] and analytically in simplifed models,[5,6] that the selection is due to transcendentally small terms in the small parameter. It was proposed in a previous publication[7] that analogous effects are present in the Saffman-Taylor problem and are responsible for the selection. This was done by a partly heuristic method similar to the one of Ref. 5, but important nonlinear effects that cannot be neglected are missed by this linear approach, as explained below. In this Letter, we give a fully nonlinear treatment of the problem, free of ary arbitrary assumption, which is inspired by the work of Kruskal and Segur[6] on the existence of needle crystals in geometric models of crystallization. Our approach to the nonlocal Saffman-Taylor model resolves the aforementioned difficulties of the linear treatment. It opens the way to a clear understanding of realistic selection phenomena in the broad area of diffusion-controlled interface motion.[8]

We explain briefly our approach before proceeding to give more details on our analysis. We are interested in understanding how an arbitrarily small interfacial tension ($k \ll 1$) selects a discrete set of finger widths out of a continuum family. McLean and Saffman's equations [see Eqs. (1a) and (1b) below] are equations for a half-finger profile. We find it convenient to introduce a new coordinate along the interface which makes apparent their symmetry with respect to reflection along the center axis of the cell. Perturbation in powers of k gives an asymptotic expansion of the finger profile which respects this natural symmetry for any finger width. The crucial point is surprisingly that this does not give enough information to decide whether a given finger exists or not. Transcendentally small terms in k (i.e., lying beyond all orders of the asymptotic expansion), whose existence is qualitatively explainable by the fact that the small parameter is in front of the highest derivative, may make the symmetric continuation of the finger incompatible with a smooth behavior at its tip. We must therefore find a method to compute such terms. Generalizing the approach of Ref. 6 to our problem, we define it in the complex plane. In the neighborhood of one of its singularities, the asymptotic expansion breaks down and small terms become large enough to be noticed and therefore are under control. Here we find a couple of singularities and in their neighborhood we obtain a rescaled problem which is different from the original one and independent of the small parameter. It gives to leading order in its asymptotic expansion the effects that were previously beyond all orders. This internal region contains therefore the whole selection mechanism. It gives, for example, the new prediction that the width of selected fingers goes to $\frac{1}{2}$ as $k^{2/3}$ when $k \to 0$, because $a = \alpha^{3/2}/k$ is the only parameter in the internal problem. We should emphasize that, in contrast to Ref. 6, our internal problem still depends on the dimensionless quantity a, which appears as a nonlinear eigenvalue of an otherwise parameterless problem. The magnitude of the terms that prevent the half-profile to be continued by parity into a full smooth finger is a function of a. The nullification of this function gives the searched for solvability condition and its zeros are the selected finger widths. We emphasize that this solvability condition is exact (within the limit of k going to zero, a finite). For large values of a, the inner problem itself has a WKB-type structure and its asymptotic solution is obtained. The actual magnitude of transcendental corrections at the finger tip could be computed via a WKB transmission factor from the internal region to the real axis. This is the main difference from the previously proposed linear analysis.[7] The internal region was not defined and the magnitude of transcendental terms was approximately estimated via an inhomogeneous second member in the WKB equation.

By using conformal mapping techniques, McLean and Saffman[2] have reduced the determination of the finger shape to the solution of two coupled nonlinear integrodifferential equations:

$$\ln(q) = \frac{-s}{\pi} \mathscr{P} \int_0^1 \frac{\theta(s')}{s'(s'-s)} ds', \tag{1a}$$

$$kqs \frac{d}{ds}\left(qs \frac{d\theta}{ds} \right) - q = -\cos\theta, \tag{1b}$$

with the boundary conditions $\theta(0) = 0$, $q(0) = 1$, $\theta(1) = \frac{1}{2}\pi$, and $q(1) = 0$; $\mathscr{P}$ denotes the Cauchy principal value. The parameter k is proportional to the surface tension and inversely proportional to the velocity of the finger. The variable s parametrizes a half finger only; $s = 1$ corresponds to the tip and $s = 0$ to the trailing part of the finger.

In the $k = 0$ case, the continuum of solutions discovered by Saffman and Taylor[1] is recovered: $q_0 = [(1-s)/(1+s\alpha)]^{1/2}$, $\theta_0 = \cos^{-1} q_0$, with $\alpha = (2\lambda - 1)/(1-\lambda)^2$, λ being the relative width of the finger.

Numerical analysis[2,3] suggests that the relative width of the finger decreases toward $\lambda = \frac{1}{2}$ when $k \to 0$. We thus focus on small values of α and k. We introduce the change of variables $s = \cosh^{-2} \frac{1}{2} v$, $v \in (-\infty, 0]$, as in Ref. 7. Although v is defined on $(-\infty, 0]$ only, the solutions can be extended to the whole v axis: $v < 0$ describes one half of the finger and $v > 0$ the other half. It is physically clear that the solutions $\theta(v) + \frac{1}{2}\pi$ and $q(v)$ must be odd functions of v. One can indeed extend the McLean-Saffman equations to the whole v axis and check that the determination of the shape of steady fingers is equivalent to the existence of odd solutions of these extended equations.

Let us start off with a solution of the Saffman-Taylor problem ($k = 0$), corresponding to a relative width λ, which defines a value of α, and look for a solution of the complete problem nearby. The perturbation analysis has been carried out by McLean and Saffman but does not give any indication of a selection phenomenon. There is a difficulty at the point $s_0 = -1/\alpha$, where the expansion is increasingly divergent. However, this point does not lay in the physical region, $0 < s < 1$. The existence of singularities in the complex domain generates exponentially small terms that can be computed[6] only by investigation of the neighborhood of these singularities. In the present problem, we need first to continue analytically Eqs. (1), and then to use a standard inner-outer expansion[9] in the vicinity of the singularity.

Equation (1a) is readily extended in the upper-half complex v plane:

$$\ln(q) = \frac{-s}{\pi} \int_0^1 \frac{\theta(s')}{s'(s'-s)} ds' + i\theta(s) = I(v) + i\theta(v), \tag{2}$$

where s is understood as a function of v. The closest singularities to the real axis are $v = i\pi + w_0$, where w_0 is a solution of $\sinh \frac{1}{2} w_0 = \sqrt{\alpha}$. The roots close to 0 of this equation are real for $\alpha > 0$ and imaginary for $\alpha < 0$. In the vicinity of the singularities ("inner problem"), $I(v)$ is a regular analytical function and can be represented by its Taylor series. In the limit $k \to 0$, it is enough to retain the lowest-order terms. Thus, Eq. (2) relates in a single fashion the varying parts of $q(v)$ and $\theta(v)$ near their singularities, allowing us to transform Eqs. (1a) and (1b) into a single differential equation for the inner problem. Moreover, one can check that $I(v)$ can be safely evaluated to zeroth order in k (i.e., by use of the original Saffman-Taylor solution). This leads to

$$ikw \frac{d}{dw}\left(w \frac{dq}{dw} \right) = -q_0^{-2} + q^{-2} = \alpha - \frac{w^2}{4} + q^{-2}, \tag{3}$$

where $w = v - i\pi$. Rescaling w and q as $w = 2\sqrt{|\alpha|}x$ and $q = Q/\sqrt{|\alpha|}$, one obtains

$$ix \frac{d}{dx}\left(x \frac{dQ}{dx} \right) = \alpha(\epsilon - x^2 + Q^{-2}), \tag{4}$$

with $\epsilon = \mathrm{sgn}(\alpha)$ and $a = |\alpha|^{3/2}/k$. As Eq. (4) is valid in the inner region only, we need to match its solution with the limit of the outer problem when w is small (although large compared to the size of the inner region). In this limit the asymptotic expansion is still valid and provides the boundary condition on Q:

$$Q \approx -1/x \text{ for } \mathrm{Re}(x) \to -\infty. \tag{5}$$

The McLean-Saffman equations have an odd solution on the real axis only if the solution of Eqs. (4) and (5) is purely imaginary on the imaginary axis. We are thus lead to a nonlinear eigenvalue problem for the parameter a whose solution gives ultimately the selection mechanism. This is the searched for exact solvability condition. In order to get more insight, we investigate the large-x behavior of the solutions of Eqs. (4) and (5). When x is large, an asymptotic expansion in powers of $1/x$ can be generated. It is expected to be correct up to small corrections. The possible form of these corrections is computed by linearization of (4) around the asymptotic value of Q, and satisfy

$$ix\frac{d}{dx}\left(x\frac{dQ_1}{dx}\right) = 2ax^3 Q_1, \tag{6}$$

where $Q = -1/x + \ldots + Q_1$. (By the ellipses we mean an asymptotic series when $|x| \to \infty$.) The only possible asymptotic behavior of Q_1 on the negative imaginary axis is

$$Q_1 = [C(a)/(ix)^{3/4}]\exp\int(-2iax')^{1/2}dx'. \tag{7}$$

It would usually be meaningless to add exponentially small terms to a truncated asymptotic series. Here, the asymptotic series is purely imaginary on the imaginary axis. Thus, its real part trivially converges to zero and can be compared to the real part of Q [the solution of Eqs. (4) and (5)] giving sense to the real part of Q_1. The integrand in Eq. (7) is real on the half imaginary axis $\mathrm{Im}(x) < 0$, so that the phase of Q_1 is given by the phase of $C(a)$. Whenever $C(a)$ is purely imaginary, we obtain an odd solution of the McLean-Saffman equations. The solvability condition is translated in the nullification of the real part of $C(a)$ whose zeros give the selected widths.

This function must in general be computed numerically by solution of the nonlinear problem [Eq. (4)] with the given boundary condition [Eq. (5)]. In the large-a limit we evaluate it as follows. We first consider the case $\alpha > 0$ (relative finger width larger than $\frac{1}{2}$). In the neighborhood of the left singularity ($x = -1 + y$, $y \ll 1$), the internal problem, Eq. (4), simplifies to

$$i\, d^2Q/dy^2 = a(2y + Q^{-2}). \tag{8}$$

As before, the small parameter $1/a$ can be eliminated by a rescaling of the function Q and variable y: $Q = a^{1/7}F$, $y = a^{-2/7}r$. In terms of the new variable and function we get

$$i\, d^2F/dr^2 = 2r + F^{-2}. \tag{9}$$

This defines a new inner region of extension $a^{-2/7} \ll 1$, for large a. Equation (9) defines the inner problem in this region. It is worth noting that we would have obtained, from the very start, the same equation if we had considered the limit k going to zero, with α fixed. We apply a second time the method of Ref. 6. As before we investigate the large-r behavior of solutions of Eq. (9) with the prescription that $F \approx -i(2r)^{-1/2}$ for $\mathrm{Re}(r) \to -\infty$. The possible exponential asymptotic correction to the algebraic series in the wedge-shaped sector $-6\pi/7 < \arg(r) < -2\pi/7$ is found to be

$$F_1 \approx \gamma r^{-3/8}\exp(2^{13/4}r^{7/4}/7). \tag{10}$$

Again γ is a constant which should be determined numerically but does not depend on a. In the outer region ($x^2 - 1 \gg a^{2/7}$), the form of transcendental corrections may be found by a WKB analysis where $1/a$ is the small parameter. We linearize Eq. (4) around its zeroth-order solution in $1/a$, $Q_0 = -1/(x^2-1)^{1/2}$, and have to solve

$$\frac{i}{a}x\frac{d}{dx}\left(x\frac{dh}{dx}\right) - 2(x^2-1)^{3/2}h = 0. \tag{11}$$

The only acceptable solution well below the segment joining $x = -1$ to $x = 1$ is

$$h = \frac{C}{(1-x^2)^{3/8}} \times \exp\int_{x_0}^{x}(-ia)^{1/2}\frac{(x'^2-1)^{3/4}}{x'}dx', \tag{12}$$

where x_0 is an arbitrary point on the negative imaginary axis. It is straightforward to check that h matches to Q_1 of Eq. (7) for large x on their common region of validity provided that C has the same phase as $C(a)$. Now, by requiring that h matches to F_1 of Eq. (10), close to $x = -1$, we find that the phase of C is deduced from the one of γ by adding the imaginary part of

$$(-2ia)^{1/2}\int_{x_0}^{-1}(x^2-1)^{3/4}x^{-1}\,dx, \tag{13}$$

which is equal to $(a/2)^{1/2}\pi$ (this is simply obtained from the integration along an infinitesimal quarter of circle below zero with a suitable choice of integration path). Finally, we get the following asymptotic solvability condition in the large-a limit for $\alpha > 0$:

$$a = 2(n + \tfrac{1}{2}\mathrm{const}/\pi)^2, \tag{14}$$

where the constant is the phase of γ defined in Eq. (10). For $\alpha < 0$, the singularities are located on the imaginary axis and the phase factor linear in $a^{1/2}$, which originates from the need of a rotation in the complex plane in the preceding situation, does not exist. Therefore there is no possible solution in this case.

In conclusion, we have shown how to remove the degeneracy of the Saffman-Taylor solution at small surface tension and how to compute selected finger widths.

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Note added.—We have now solved our nonlinear problem and the results are in agreement with known numerical results.[2,3] We intend to publish a detailed account of this work elsewhere. After this letter was sent for publication, we received a preprint from Shraiman[10] and another one from Hong and Langer[11] where approximate solvability conditions for the Saffman-Taylor problem are obtained by linear methods, the treatment essentially identical to the one given by three of us in Ref. 7.

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