# Analytic Theory of the Selection Mechanism in the Saffman-Taylor Problem

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We present an analytic approach to the problem of predicting the widths of fingers in a Hele-Shaw cell. Our analysis is based on the WKB technique developed recently for dealing with the effects of surface tension in the problem of dendritic solidification. We find that the relation between the dimensionless width $\lambda$ and the dimensionless group of parameters containing the surface tension, $\nu$, has the form $\lambda - \frac{1}{2} \sim \nu^{2/3}$ in the limit of small $\nu$.

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The Saffman-Taylor[1,2] problem continues to play a central role in current theories of pattern formation in nonlinear dissipative systems. This problem of predicting the shape of the fluid finger that forms in a Hele-Shaw cell is similar in many respects to the problem of predicting the shape and speed of a dendritic crystal growing in an undercooled melt. Pattern selection in both of these problems turns out to be controlled by surface tension, an ostensibly small but singular perturbation which converts an equation with a continuous family of steady-state solutions into one for which such solutions exist only for a discrete set of values of some parameter, for example, the finger width or the growth rate. Among those configurations which satisfy such a solvability condition, there generally exists just one which can be identified as the dynamically selected state of the system.

Almost all of the specific applications[3-6] of this solvability principle that have been carried out so far have required numerical solutions of the singular equations. It seems to us that there is an obvious need for a more analytic approach that might provide deeper understanding of the roles played by the various parameters that enter into these theories. Some progress[7] along these lines has been made recently in studies of local models[8-11] of crystal growth where there are no long-range interactions between distant parts of the solidification front or between the crystal and the walls of the container.[12] The Saffman-Taylor problem, on the other hand, is fully nonlocal—so much so that the width of the finger is determined by the width of the channel in which it is moving, and the directional information provided by the walls of the channel apparently plays a role similar to that of crystalline anisotropy in the dendrite problem. We therefore find it particularly interesting to discover that the Saffman-Taylor problem can be solved almost completely by analytic techniques which are much the same as those which have worked so well for the local models of solidification. Our purpose in this note is to summarize briefly the strategy that leads us to an asymptotic formula for the width of the viscous finger in the limit of small but nonzero surface tension. We promise a more complete report in the near future.

Our starting point is the nonlinear integrodifferential equation of McClean and Saffman[3] (hereafter referred to as MS). The system of interest is an effectively two-dimensional channel of width $2a$ and thickness $b \ll a$ along which a fluid of viscosity $\mu$ is being driven at velocity $V$ by an immiscible second fluid of relatively negligible viscosity. Both fluids are incompressible. The steady-state configuration of this system is observed experimentally to be one in which the inviscid driving fluid forms a finger of width $2\lambda a$ along the center of the channel; and the problem to be solved is to compute $\lambda$ as a function of $a$, $b$, $V$, $\mu$, and the interfacial surface tension $\gamma$.

The MS equations implicitly specify the shape of this finger by determining $\theta$, the angle of orientation of its surface, as a function of a real parameter $s$. In the notation of MS, $s=1$ at the tip of the finger and $s \to 0$ infinitely far back along one side; $\theta(1) = \frac{1}{2}\pi$ at the tip and $\theta(0) = 0$. The equation for $\theta(s)$ is

$$\nu q s \frac{d}{ds}\left(q s \frac{d\theta}{ds}\right) = q - \cos\theta, \tag{1}$$

where the function $q(s)$ is defined by

$$\ln q(s) = -\frac{s}{\pi}\mathcal{P}\int_0^1 \frac{\theta(s')}{s'(s'-s)}\,ds', \tag{2}$$

and

$$\nu = \gamma b^2\pi^2/12\mu V a^2(1-\lambda)^2 \tag{3}$$

is the small dimensionless parameter which contains the surface tension. The function $q(s)$ is proportional to the tangential velocity of the fluid at the interface and, accordingly, must satisfy the boundary conditions $q(0)=1$, $q(1)=0$. The symbol $\mathcal{P}$ denotes the Cauchy principal value. The integral in (2) is indicative of the nonlocality of this hydrodynamic system.

When $\nu = 0$, (1) and (2) can be solved explicitly to yield[3]

$$q_0(s) = \cos\theta_0(s) = \left(\frac{1-s}{1+\alpha s}\right)^{1/2}, \tag{4}$$

where $\alpha=(2\lambda-1)/(\lambda-1)^2$. Equation (4) is automatically consistent with the boundary conditions for any $\lambda$, and thus this solution fails to determine the selected $\lambda$. Our next step, still formally following MS, is to let $\nu$ be small but nonzero and linearize (1) and (2) in the neighborhood of the solution (4).[13,14] We define $\theta(s)=\theta_0(s)+\nu\theta_1(s)$, and obtain an inhomogeneous linear equation of the form

$$\nu\frac{d^2\theta_1}{ds^2}+\nu P(s)\frac{d\theta_1}{ds}+Q(s)\theta_1+H(s)\mathscr{P}\int_0^1\frac{\theta_1(s')ds'}{s'-s}=R(s), \tag{5}$$

where

$$P(s)=\frac{1}{s}-\frac{(1+\alpha)}{2(1-s)(1+\alpha s)}, \tag{6}$$

$$Q(s)=-\frac{(1+\alpha)^{1/2}(1+\alpha s)^{1/2}}{s^{3/2}(1-s)}, \tag{7}$$

$$R(s)=\frac{(1+\alpha)^{1/2}}{2s^{1/2}(1-s)^{1/2}(1+\alpha s)}\left[\frac{1}{2s}-\frac{3\alpha}{2(1+\alpha s)}\right], \tag{8}$$

and

$$H(s)=(1+\alpha s)^{1/2}/\pi s^2(1-s)^{1/2}. \tag{9}$$

Relevant boundary conditions are $\theta_1(0)=\theta_1(1)=0$. Note that, if we set $\nu=0$ in (5), the remaining equation is identical to (29) of MS. We depart from MS at this point by noticing that the terms proportional to $\nu$ in (5) are derivatives and therefore constitute a singular perturbation which cannot necessarily be discarded without missing essential features of the solution.

In order to construct $\nu$-dependent solutions of (5), we first convert it into a purely differential equation by defining $\phi$ to be a function of the complex variable $z$ such that

$$\phi(z)=\int_0^1 ds\,\theta_1(s)/(s-z), \tag{10}$$

and

$$\theta_1(s)=\pi^{-1}\lim_{\epsilon\to 0}\mathrm{Im}\phi(s+i\epsilon). \tag{11}$$

We complete the specification of $\phi$ by letting it be a solution of

$$\nu\frac{d^2\phi}{dz^2}+\nu P\frac{d\phi}{dz}+\tilde{Q}(s)\phi=i\pi R, \tag{12}$$

with

$$\tilde{Q}(z)=Q(z)+i\pi H(z)=-\frac{1+\alpha z}{z^2(z-1)}e^{i\tilde{\theta}(z)}, \tag{13}$$

where $\tilde{\theta}(z)=\theta_0(z)+\frac{1}{2}\pi$. The imaginary part of (12) is identical to (5). Note that the physical values of $\phi$ occur on the branch cut between $z=0$ and $z=1$, and that $\tilde{\theta}$ is the natural angular variable which vanishes at the tip, $z=1$, and changes from $+\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$ as $z$ moves clockwise all the way around this cut starting just above it at $z=+i\delta$ $(\delta\to 0)$ and returning to $z=-i\delta$.

Our next step is to construct a solution of (12) by a WKB method similar to that used for the local models of solidification.[7,12,14] The homogeneous solutions are found to be

$$\phi_\pm^H(z)\approx\exp[\pm(1/\sqrt{\nu})\psi(z)-\tfrac{1}{4}i\tilde{\theta}(z)], \tag{14}$$

where

$$\psi(z)=\int_1^z\frac{dz'}{z'}\left(\frac{1+\alpha z'}{z'-1}\right)^{1/2}e^{i\tilde{\theta}(z')/2}. \tag{15}$$

Examination of these formulas reveals that neither $\phi_+^H$ nor $\phi_-^H$, by itself, is an acceptable component of a solution of (12). The function $\phi_+^H$ is inconsistent with (10) because it diverges at $|z|\to\infty$; and $\phi_-^H$ diverges as $z\to\pm i0$, that is, as $\tilde{\theta}\to\pm\frac{1}{2}\pi$. However, we can use these two homogeneous solutions to construct a particular solution that is well behaved as $|z|\to\infty$, and, say, near $\tilde{\theta}=+\frac{1}{2}\pi$. This solution has the form

$$\phi^P(z)\cong(\pi/\sqrt{\nu})\int_1^z dz'\,f(z')[\phi_+^H(z)\phi_-^H(z')-\phi_-^H(z)\phi_+^H(z')]+A^P\phi_-^H(z), \tag{16}$$

where

$$f(z)=\tfrac{1}{2}z[(1-z)/(1+\alpha z)]^{1/2}R(z) \tag{17}$$

and $A^P$ is a constant which is chosen to cancel the divergence at $z=+i\delta\to 0$. That is,

$$A^P=-(\pi/\sqrt{\nu})\int_{i\delta}^1 dz\,f(z)\phi_+^H(z). \tag{18}$$

The problem with (16) is that the resulting $\theta_1$ does not vanish at the tip, $s=1$. A related difficulty is that, if we analytically continue $\phi^P(z)$ from $z=0$, above the cut, around $z=1$, and back to $z=0$ below the cut, we again find a divergence. The situation is directly analogous to what happens in the dendrite problem where solutions that are well behaved far from the tip

on one side of the needle crystal do not connect analytically to well-behaved solutions on the other side except for special values of the growth velocity. The appropriate procedure for identifying solutions in the present case is simply to use (11) and set $\theta_1(1)=0$; that is, $\mathrm{Im}\phi^P(1)=\mathrm{Im}A^P=0$ is the solvability condition that determines special values of $\lambda$ for which physically acceptable solutions occur.

To implement the above procedure, it is useful to transform the variable $\eta=\tan\tilde{\theta}$, in terms of which $z=(1+\beta^2\eta^2)^{-1}$, where $\beta=(1+\alpha)^{1/2}=\lambda/(1-\lambda)$. We then find that

$$\psi(z)=\tilde{\psi}(\eta)$$
$$=2i\beta^2\int_0^\eta d\eta'\frac{(1+i\eta')^{3/4}(1-i\eta')^{1/4}}{1+\beta^2\eta'^2} \qquad (19)$$

and, after some further algebra, can write the solvability condition in the form

$$\theta_1(1)=\frac{1}{\sqrt{\nu}}\int_{-\infty}^{\infty}d\eta\,F(\eta)\exp\left[\frac{\tilde{\psi}(\eta)}{\sqrt{\nu}}\right]=0, \qquad (20)$$

where

$$F(\eta)=\frac{\eta[6\alpha-\beta^2(1+\eta^2)]}{4(1+\beta^2\eta^2)(1+\eta^2)^{3/2}}\left[\frac{1-i\eta}{1+i\eta}\right]^{1/8} \qquad (21)$$

The integral in (20) can be evaluated numerically; but its most interesting properties can be obtained by a steepest-descent approximation. From (19) we see that $\tilde{\psi}$ has points of stationary phase at $\eta=\pm i$. It also has logarithmic branch points at $\eta=\pm i\beta^{-1}\cong\pm i[1-4(\lambda-\frac{1}{2})]$ which, for $\lambda>\frac{1}{2}$, lie closer to the real axis than the stationary points at $\pm i$. For small $\nu$, the dominant contribution to (20) comes from the neighborhood of $\eta=+i$. For $\lambda>\frac{1}{2}$, however, the stationary point $\eta=+i$ must be interpreted as a complex conjugate pair of points on either side of a branch cut running from $\eta=i\beta^{-1}$ to $+i\infty$. Accordingly, the path of steepest descent must include a section which goes from $\eta=i-\delta$ $(\delta\to+0)$, around the branch point at $i\beta^{-1}$, and back to $i+\delta$. If this section of the contour is long enough to contain many oscillations of the integrand, then $\theta_1(1)$ itself will be an oscillating function of $\lambda$. On the other hand, if $\lambda$ is close enough to $\frac{1}{2}$ (or less than $\frac{1}{2}$), then the branch point will be too close to the stationary point (or above it), so that this analytic structure will not be resolved by the integrand and no oscillations will occur in $\theta_1(1)$. This is precisely the same analytic mechanism that produces zeros in the solvability function for the local models of dendritic solidification.[14] The quantity $\beta^2-1$ turns out to be a direct mathematical analog of the crystalline anisotropy.

To make this description more precise, let $\eta=i+\omega$ and use (19) to compute the form of $\tilde{\psi}$ for $|\omega|\ll 1$:

$$\Delta\tilde{\psi}(\omega)=\tilde{\psi}(\eta)-\tilde{\psi}(i)\approx\text{const}\times\frac{\beta^2\omega^{7/4}}{1-\beta^2}. \qquad (22)$$

The crossover from oscillating to smooth behavior will occur when $\Delta\tilde{\psi}(\omega)/\sqrt{\nu}$ is of order unity for values of $|\omega|$ equal to the effective distance along the contour between the stationary points, that is, for $|\omega|\sim 1-\beta^{-1}\sim\lambda-\frac{1}{2}$. Inserting this condition into (22), we find $\lambda-\frac{1}{2}\sim\nu^{2/3}$. The final step in the development is to identify this crossover condition as the actual selection criterion. In this case, as with the dendrites,[14] we argue that the solution with the smallest $\lambda$ (or the largest $\nu$ at fixed $\lambda$) is most likely to be dynamically stable—that broader fingers with flatter fronts would be subject to the same instabilities of the flat surface as those which produced fingers in the first place. We should emphasize, however, that none of this analysis touches directly on questions of dynamical stability.

In summary, we find that the solvability condition for the Saffman-Taylor problem has a discrete set of solutions (infinitely many in the limit of small $\nu$ and fixed $\lambda$). The physically selected finger is predicted to have a width $\lambda$ such that $\lambda-\frac{1}{2}$ is proportional to $\nu^{2/3}$ in the limit of small $\nu$, a result which seems to be consistent with numerical data.[3,4] Perhaps most important, from a more general point of view, is that the detailed mathematical structure of the solvability mechanism for this nonlocal problem is essentially the same as that which has been found for local models of solidification.

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[13]This linearization is mathematically the most subtle aspect of our analysis. M. Kruskal and H. Segur (unpublished) have shown in a special example that the solvability function analogous to $\theta_1(1)$ is given correctly by this method up to a constant prefactor which is irrelevant for our purposes. More details of this analysis are given by J. S. Langer and D. C. Hong, to be published.

[14]Langer and Hong, Ref. 13.