Velocity Selection and the Saffman-Taylor Problem

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A new approach to the velocity selection problem is presented. It establishes a relation between the existence of propagating steady states as a function of a parameter and a certain inhomogeneous linear problem, the solvability of which determines the set of allowed velocities. The method is illustrated on the example of a geometrical model of solidification. It is used to explain analytically the fact that at large velocity the Saffman-Taylor "fingers" have width close to $\frac{1}{2}$ and to predict the scaling exponent for the dependence of the finger width on velocity.

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The study of patterns formed in nonequilibrium systems is often confounded by the problem that the theory seems to allow for a continuous family of steady-state solutions, while a unique state is reproducibly observed in the experiment. Such is the situation with the growth of dendrites,[1] propagation of fingers of one viscous fluid into another[2-4] (the Saffman-Taylor problem), and directional solidification.[5]

Saffman and Taylor[2] in 1958 studied the displacement of a viscous fluid in a thin cell by air. Their theory, which neglected the surface tension, described a continuous family of steady propagating finger solutions, suggesting that fingers occupying an arbitrary fraction λ of the width of the cell can exist. For a fixed velocity of displaced fluid, V, which is controlled in the experiment (the dimensionless control parameter[4] is proportional to $\mu V/\sigma$, with σ being the surface tension and μ the dynamic viscosity), the fingers would propagate with different velocities V/λ. However, the experiment reported in the same paper observed unique steady states with the width of the fingers λ approaching $\frac{1}{2}$ for large velocities V. Thus the problem was to explain the selection of that particular state. The importance of the surface tension in the problem was recognized immediately: Without it the dynamics of the interface is unstable. The steady-state equations with surface tension included were solved numerically by McLean and Saffman[6] and Vanden-Broeck[7] who found that the continuous family of solutions broke down to a discrete set. However, the perturbation theory for small surface tension failed to show this "selection" and no satisfactory analytic understanding emerged.

Kessler, Levine, and Koplik[8] and Ben-Jacob *et al.*[9] proposed on the basis of their studies of local models of solidification (a geometric and a boundary-layer model, respectively) that the "velocity selection" problem is generically resolved by solvability conditions arising from the singular effects of the surface tension. Their method followed that of Vanden-Broeck[7] and consisted of relaxing one of the boundary conditions for the steady-state equation, numerically integrating the equation for different values of the parameter, and then selecting those values of the parameter for which all the boundary conditions were satisfied. An important step in the analytic understanding of the problem was made by Langer[10] who, using a WKB approximation on the example of a simple geometrical model, exposed the delicate nonperturbative nature of the solvability condition. The minimal geometric model was also studied in the recent work of Kruskal and Segur[11] and Dashen *et al.*[12] using different approximations. However, the conclusions drawn from the analysis of local models[8,9] are still contested,[13] and it is most important to be able to address the more physical, *nonlocal* problems directly.

In this paper a new approach, alternative to the "matching" method described above, is introduced. The advantage of the present formulation is that it puts the question of the existence of the family of solutions into a standard form familiar from perturbation theory. Combined with the WKB approximation[10,11,14] it provides a powerful tool for the analysis of both local and nonlocal models. Below, after formulating the method, I illustrate it on the example of a "minimal" geometric model, for which the dependence of the selected velocity on the anisotropy is obtained. The approach is then applied to the Saffman-Taylor problem and allows us to show that only the $\lambda=\frac{1}{2}$ finger exists in the limit of vanishing surface tension. For small surface tension a discrete set of solutions is found. The width of the "fingers" approaches $\frac{1}{2}$ with exponent $-\frac{2}{3}$ as the control parameter goes to infinity. This scaling agrees well with the numerical results.[4,7,15].

Instead of solving the nonlinear steady-state equation for different parameter values one can ask whether, given a solution at one set of the parameters, there is also a solution nearby. One then looks for a mode corresponding to the infinitesimal change of parameters, which quite generally entails solving an inhomogeneous *linear* problem. Formally, let the steady-state shape be described by $X(s)$ determined by some equation containing a set of parameters γ (e.g., σ and λ for the Saffman-Taylor problem): $F_\gamma\{X\}=0$. Assume that X_γ solves it for some γ and define an associated linear operator $\mathbf{K}\equiv\delta F_\gamma/\delta X$. If a continuous family of solutions exists, there is a mode $\chi=\partial_\gamma X_\gamma$, corresponding to an infinitesimal change of γ along some *line* in the parameter space, satisfying

$$\mathbf{K}\chi=-\partial_\gamma F_\gamma|_{X_\gamma}. \tag{1}$$

Conversely, a continuous family passes through the point γ if and only if the "parametric" mode χ exists at that point. This requires solvability of an inhomogeneous linear problem given by Eq. (1). If the linear operator has a zero eigenmode ξ, $\mathbf{K}\xi=0$, there is a solvability condition[14] requiring the inhomogeneous term of Eq. (1) to be orthogonal to the zero eigenmode η of the adjoint operator $\mathbf{K}^\dagger$:

$$\langle\eta,\partial_\gamma F_\gamma|_{X_\gamma}\rangle=0. \tag{2}$$

The orthogonality condition of Eq. (2) will in general define surfaces in the parameter space for which solutions exist. The intersections of these surfaces with lines corresponding to the free parameter will determine selection.

I now illustrate the method for the example of the minimal geometric model[10,12] of interface dynamics. In this model the normal velocity v_n of the interface (parametrized by the arc length s) is determined by the local curvature κ and the angle of the normal θ:

$$v_n=(\kappa+\sigma\,\partial_s^2\kappa)(1+\bar\epsilon\cos2\theta), \tag{3}$$

where σ is the surface tension, $\bar\epsilon$ is the coefficient of twofold anisotropy, and $\kappa=\partial_s\theta(s)$. A steady translation of the interface with velocity v is described by $v_n=v\cos\theta$, so that the steady-state equation has the form

$$\sigma\,\partial_s^3\theta(s)+\partial_s\theta(s)+[v/(1-\bar\epsilon)]\,U_\epsilon(\theta(s))=0, \tag{4}$$

with $U_\epsilon(\theta)\equiv\cos\theta/(1+\epsilon^2\cos^2\theta)$ and $\epsilon^2\equiv2\bar\epsilon/(1-\bar\epsilon)$ $<<1$. In the absence of surface tension, $\sigma=0$, Eq. (4) can be integrated, yielding a family of solutions $\theta_0(vs(1-\bar\epsilon)^{-1})$ related by a scale transformation.

We now address the full problem by constructing the linear operator

$$\mathbf{K}_\nu=\nu^2\partial_\tau^3+\partial_\tau+U_\epsilon'(\theta_\nu(\tau)), \tag{5}$$

where a reduced velocity $\nu\equiv v\sqrt{\sigma}/(1-\bar\epsilon)<<1$ is defined and $\tau\equiv vs/(1-\bar\epsilon)$. If $\theta_\nu(\tau)$ is a solution of Eq. (4), then, since Eq. (4) does not involve s explicitly, $\partial_\tau\theta_\nu(\tau)$ is a zero eigenmode of $\mathbf{K}$. Thus, there will be a nontrivial solvability condition! The physical formulation of the problem requires the eigenfunctions to vanish at infinity. We shall require square integrability and use the standard Hilbert-space definition of a scalar product. The adjoint operator $\mathbf{K}^\dagger$ is then

$$\mathbf{K}^\dagger\equiv-\nu^2\partial_\tau^3-\partial_\tau+U_\epsilon'(\theta_\nu(\tau)). \tag{6}$$

We now look for a zero eigenmode of the adjoint, $\mathbf{K}^\dagger\eta=0$, using a WKB approximation[14] in the spirit of Langer[10]: $\eta(\tau)=A(\tau)\cos(\nu^{-1}\tau)$. The fast phase factor balances the derivative terms of the equation while the slow amplitude obeys $\partial_\tau\ln A(\tau)=-\frac{1}{2}U_\epsilon'(\theta_\nu(\tau))$ and for $\nu<<1$ one obtains

$$\eta(\tau)=A_0[U_\epsilon(\theta_0(\tau))]^{1/2}\cos(\nu^{-1}\tau)+O(\nu^2). \tag{7}$$

For the "parametric" mode corresponding to the change $(d\nu,d\epsilon)$ in the two relevant parameters the solvability condition, Eq. (2), for $\nu\to0$ requires that

$$S=\langle\eta,\partial_\tau^3\theta_0\rangle+\frac{\epsilon}{\nu}\frac{d\epsilon}{d\nu}\langle\eta,\partial_{\epsilon^2}U_\epsilon(\theta_0)\rangle=0, \tag{8}$$

where the angular brackets stand for the integral over τ. The $\sigma=0$ solution of Eq. (4), $\theta_0(\tau)$, is given implicitly by $\tau=x+\epsilon^2\tanh x$ and $\tanh x=\sin\theta_0$. Then

$$-\partial_\tau\theta_0=U_\epsilon(\theta_0(\tau))$$
$$=\cosh x(\tau)[\cosh^2x(\tau)+\epsilon^2]^{-1}.$$

In order to evaluate the integrals in Eq. (8) asymptotically[14] for $\nu<<1$ we study the analytic structure of the integrands. The dominant contribution comes from the closest pair of branch points in the upper half-plane which for $\epsilon^2<<1$ are at $\tau=\pm2\epsilon+i\pi/2$ [at these points $\cosh^2x(\tau)+\epsilon^2=0$]. In their vicinity the integrands can be simplified by the introduction of $2\epsilon\zeta\equiv s-i\pi/2$, $|\epsilon\zeta|<<1$, and expansion of the hyperbolic functions. Explicitly one finds that

$$S=\frac{e^{-\pi/2\nu}}{2^{3/2}\epsilon^{5/2}}\left[\frac{3}{2}I_{11/4}\left(\frac{\epsilon}{\nu}\right)+I_{7/4}\left(\frac{\epsilon}{\nu}\right)+\frac{\epsilon}{\nu}\frac{d\epsilon}{d\nu}I_{3/4}\left(\frac{\epsilon}{\nu}\right)\right], \tag{9}$$

with

$$I_\gamma(z)=\mathrm{Re}\left\{e^{-i\pi\gamma}\int_{-\infty-i0+}^{\infty-i0+}d\zeta[\zeta^2-1]^{-\gamma}e^{i2z\zeta}\right\}. \tag{10}$$

The bracketed expression in Eq. (9) depends only on the ratio $r\equiv\epsilon/\nu$ so that the solvability condition has a form $f_1(r)+[r+\nu(dr/d\nu)]f_2(r)=0$ and in the limit $\nu\to0$ requires that $f_1(r)+rf_2(r)=0$. For $r>>1$ the integrals are readily evaluated by the steepest-descent method yielding the asymptotic[16] form of the solvability condition: $\cos(2\epsilon/\nu-3\pi/8)=0$. The latter has a countable infinity of solutions which in terms of the original parameters of the problem define a set of "allowed" velocities: $v_k^2\approx8\bar\epsilon/\pi^2\sigma k^2$ for integer $k>>1$. This result fits well with what is known about the model and proves the conjecture about the asymptotic behavior of this set made by Dashen *et al.*[12]

The above example illustrates well most of the important points of the method. The solvability condition is associated with the zero mode, $\partial_\tau\theta_0$, which appears because of the reparametrization invariance of the interface (it reflects the freedom left after specification of a particular parametrization) and is thus typical of the whole class of problems. A peculiar feature of the problem is that while $\partial_\tau \theta_\nu$ is a "slow" function of τ, the adjoint zero mode η is not! It has a "fast" phase factor varying with τ/ν. This behavior is a consequence of the singular nature of the small-surface-tension limit and is responsible for the nonperturbative (exponentially small) form of the term producing the solvability condition.

Let us now discuss the application of the method to the Saffman-Taylor problem.[2] Unlike the geometrical model the steady-state equation for the interface of two fluids is *integro*differential. It is determined implicitly by the constraint that the velocity potential (obeying the Laplace equation) satisfies *two* boundary conditions on the interface and can be derived by conformal mapping techniques, which were reviewed in Ref. 4. Below, I shall only sketch the analysis of the solvability of this steady-state equation for small surface tension (large values of the control parameter[4]), leaving the details of the calculation to be published elsewhere.[17]

A steady propagating finger occupying a fraction $\lambda = (1+\alpha)^{-1}$ of the channel width satisfies the equation

$$\nu^2 \partial_\phi \mathbf{H}_0\{\kappa\} - (1+\alpha) e^{g(\phi)} \cos\theta(\phi) + 1 = 0, \tag{11}$$

with $\nu^2 \sim \sigma/\mu V << 1$ being the dimensionless surface tension, θ the angle of the normal, and $g(\phi) = \ln(ds/d\phi)$ with s the arc length. The parametrization ϕ, $-\pi < \phi < \pi$, is determined by the condition that $g(\phi) + i\theta(\phi)$ is analytic outside the unit circle $\exp(i\phi)$, which means that $g = -\mathbf{H}_0\{\theta\}$, $\mathbf{H}_0\{\ldots\}$ being the Hilbert transform on the circle. Finally, $\kappa = \exp(-g)\partial_\phi \theta$ is the curvature.

For zero surface tension, $\nu = 0$, there is a family of steady states: $\exp(g_0 + i\theta_0) = \lambda + i(1-\lambda)\tan(\phi/2)$ corresponding to fingers of width λ moving with velocity V/λ. To study the problem for $\nu << 1$ we construct the linear operator by expanding in $\xi = \delta\theta$ and $\delta g = -\mathbf{H}\{\xi\}$:

$$\mathbf{K}\xi = \tfrac{1}{4}\nu^2(1+\tau^2)\partial_\tau \mathbf{H}\{\gamma_\alpha \partial_\tau \xi + \kappa_\alpha \mathbf{H}\{\xi\}\} + \mathbf{H}\{\xi\} + \alpha\tau\xi(\tau), \tag{12}$$

with a new variable $\tau = \tan(\phi/2)$, $\mathbf{H}$ the Hilbert transform on the real line, $\gamma_\alpha(\tau) \equiv (1+\alpha)(1+\tau^2)(1+\alpha^2\tau^2)^{-1/2}$, and $\kappa_\alpha(\tau) = \alpha(1+\alpha)(1+\tau^2)(1+\alpha^2\tau^2)^{-3/2}$.

As before, the zero mode of $\mathbf{K}$ is readily found: $\xi(\tau(\phi)) = \partial_\phi \theta_0$. This implies the existence of the solvability condition and one has to proceed by defining the adjoint operator and looking for its zero mode η. It can be found by use of the WKB approximation[18] and has the form $\eta(\tau) = \exp[\nu^{-1}\Psi(\tau)]$, with

$$\frac{d\Psi}{d\tau} = -\frac{\sqrt{2}}{(1+\alpha)^{1/2}} \frac{\alpha\tau - i[1+(1+\alpha^2\tau^2)^{1/2}]}{(1+\tau^2)[1+(1+\alpha^2\tau^2)^{-1/2}]^{1/2}} + O(\nu). \tag{13}$$

The function Ψ has a logarithmic singularity at $s = \pm i$, with a prefactor $\epsilon^{3/4}/\sqrt{2}$ for $0 < \epsilon \equiv \lambda - \frac{1}{2} << 1$. This singularity dominates over the saddle point at $s = \pm i\alpha^{-1}$, when the steepest-descent method is used to evaluate the integrals appearing in the solvability condition which has a form similar to that in Eq. (8). The resulting condition (asymptotic for $\nu^{-1}\epsilon^{3/4} >> 1$)

$$\cos(\pi\epsilon^{3/4}/\sqrt{2}\nu + \text{const}) = 0 \tag{14}$$

is the main result of the paper: For small values of dimensionless surface tension or, equivalently, for large velocity, there is a discrete set of finger solutions with $2\lambda - 1 \sim \nu^{4/3} \sim (\sigma/\mu V)^{2/3}$. This scaling fits well the results of earlier numerical simulations.[4,7,15] Experimentally, fast fingers do have width very close to $\frac{1}{2}$. Unfortunately, the instability that sets in at large velocity[3,4] limits experimental study of the scaling.

In conclusion, this paper presented an approach to the velocity selection, which allowed us to understand and determine the scaling for the Saffman-Taylor problem for small surface tension analytically. It would be useful to develop simple solvability arguments based on counting modes and boundary conditions. Another very interesting question is understanding why the solvability condition for the steady state controls the dynamical process of pattern selection. The present approach may again be useful: The parametric mode is relevant dynamically.

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[16]The validity of the WKB approximation is a subtle question. While it provides a uniformly convergent approximation for η on the real axis, the evaluation of the scalar product requires a uniform approximation in the upper half-plane. One can show (Ref. 11 and B. I. Shraiman, to be published) that the vicinity of the branch points represents an "inner" problem (Ref. 14) for Eq. (6) which in that region depends on the ratio ϵ/ν only, and does not have a small parameter. For $\epsilon/\nu >> 1$, however, the "outer" WKB expansion is valid even in the inner region, assuring the correct asymptotic behavior of the solvability condition. The WKB method is also sufficient for determination of the structure and the scaling behavior of the set of allowed velocities.

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[18]The singularities of η in the upper half-plane turn out to have an exponentially small prefactor, so that $\mathbf{H}\{\eta\} = -\eta + O(\exp(-\nu^{-1}))$, which simplifies the calculation!