Current-Mass Ratios of the Light Quarks

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We investigate the current-mass ratios of the light quarks by fitting the squares of meson masses to second order in chiral-symmetry breaking, determining corrections to Weinberg's first-order values: $m_u/m_d=0.56$, $m_s/m_d=20.1$. We find that to this order, m_s/m_d is a known function of m_u/m_d. The values of the quark-mass ratios can be constrained by limiting the size of second-order corrections to the squares of meson masses. We find that for specific values of presently unmeasured phenomenological parameters one can have a massless u quark. In that case 30% of the squares of meson masses arise from operators second order in chiral-symmetry breaking.

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In spite of the wealth of hadronic data, it remains impossible to extract reliable values for the current masses of the u, d, and s quarks. This well known difficulty arises from the fact that these quarks have strong interactions characterized by a mass scale that is much greater than their mass parameters in the QCD Lagrangean. However, as Weinberg has shown us, one can get a handle on the mass *ratios* of these quarks by using $SU(3)\otimes SU(3)$ current algebra to relate these ratios to the measured pseudoscalar-meson masses. He finds $m_u/m_d=0.56$, $m_s/m_d=20.1$.[1] But how reliable are these values? The joke of the matter is that while the mass of the s quark is too small compared to the hadronic scale to be measured directly, it is too large for $SU(3)\otimes SU(3)$ current algebra to be very reliable at lowest order in chiral-symmetry breaking. Furthermore, one loses predictive power in going to higher order in chiral perturbation theory. Thus any attempt to investigate corrections to Weinberg's mass ratios is going to involve more effrontery than artistry.

This Letter is precisely such an attempt. We work in the context of an $SU(3)\otimes SU(3)$ chiral Lagrangean and calculate the meson masses including operators up to second order in the quark mass matrix M or the electric charge Q. We also include logarithmic contributions of comparable magnitude from chiral loops. Since there are more operators (and hence more incalculable strong-interaction coefficients) than there are meson masses, one cannot calculate the quark-mass ratios in the straightforward manner that Weinberg used. One possible approach is to parametrize the chiral Lagrangean in such a way that the unknown strong-interaction coefficients are all dimensionless numbers expected to be ~ 1. This may be done by the dimensional-analysis argument of Weinberg and of Manohar and Georgi.[2] Then one can scan values of the quark-mass ratios which allow one to fit the meson masses, consistent with the prejudice that these coefficients are truly ~ 1, and not ~ 10, for example.

We have chosen a similar, slightly more phenomenological approach. We bound the second-order contributions to the meson masses, rather than bounding the individual, unmeasurable coefficients of the second-order operators. This lets one make use of the observation that most lowest-order $SU(3)\otimes SU(3)$ chiral perturbation calculations disagree with experiments by about 25%. One might then "reasonably" expect to find that second-order terms contribute 25% to the squares of physical meson masses.

The results of such a computation are shown in Fig. 1. As explained below, m_s/m_d is found to be a function of m_u/m_d, the physical meson masses, and f_π; hence one finds a curve rather than a shaded region in a plot of m_s/m_d vs m_u/m_d. We have marked the nested segments of the curve which are allowed when one

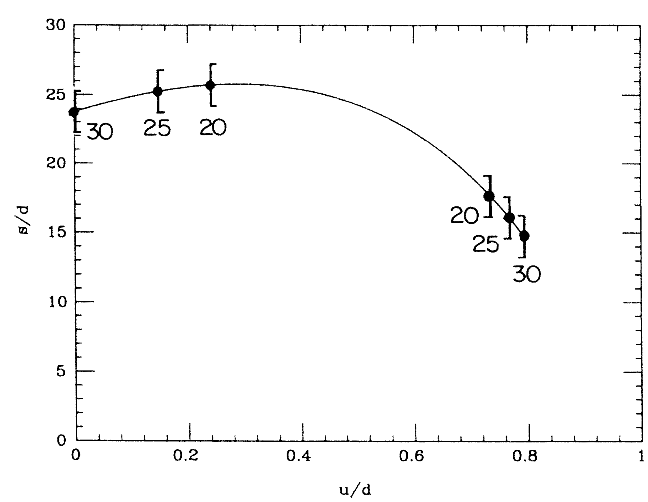

FIG. 1. A plot of the current-quark-mass ratios m_s/m_d vs m_u/m_d consistent with the pseudoscalar-meson masses to second order in chiral perturbation theory, including logarithmic effects. Allowed values are constrained to lie on the curve. By restricting second-order contributions to the squared meson masses to be less than 20%, 25%, or 30% of the lowest-order values, we can exclude regions of the curve outside the brackets so marked.

limits the second-order contributions to the squares of meson masses to be 20%, 25%, and 30% of their lowest-order values. The reason why there is a *range* of allowed values for m_u/m_d is because we do not know the coefficients of the higher-dimension operators. These coefficients could, in principle, be determined by analysis of more accurate pion and kaon scattering data than are presently available. The values of these coefficients are determined by QCD dynamics. Note that a massless up quark is consistent with chiral perturbation theory if one allows up to 30% of the squares of meson masses to come from second-order effects. Such a possibility provides an economical solution to the strong *CP* problem, as has been often discussed and often discarded in the literature.

In the following we discuss the details of our calculation, and explain precisely what we mean by the percentage corrections mentioned above.

We parametrize the $SU(3)\otimes SU(3)$ chiral Lagrangean in terms of the field $\Sigma=\exp(2i\pi_a T_a/f)$, where π_a is the pseudoscalar octet, and to leading order $f=f_\pi=93$ MeV. Σ transforms as a $(3,3^*)$ under $SU(3)\otimes SU(3)$. There is explicit symmetry breaking due to the bare quark masses and electromagnetic couplings, which are taken to transform as $M=(3,3^*)$ and $Q=(8,1)\oplus(1,8)$, respectively. The most general chiral Lagrangean one can write to second order in M, Q, and ∂^2 [ignoring operators which begin at $O(\pi^4)$] is given by

$$\begin{aligned}\mathcal{L}=&(f^2/4)\,\mathrm{Tr}(\partial_\mu\Sigma\,\partial^\mu\Sigma^\dagger)+(f^2/2)\,\mathrm{Tr}(\mu M\Sigma)+\mathrm{H.c.}+c_1(f^2/2\Lambda^2)\,|\mathrm{Tr}(\mu M\Sigma)|^2\\&+c_2(f^2/2\Lambda^2)[\mathrm{Tr}(\mu M\Sigma)]^2+\mathrm{H.c.}+c_3(f^2/2\Lambda^2)\,\mathrm{Tr}[(\mu M\Sigma)^2]+\mathrm{H.c.}\\&+c_4(\alpha/4\pi)(f^2\Lambda^2/2)\mathrm{Tr}(Q\Sigma Q\Sigma^\dagger)+c_5(f^2/2\Lambda^2)\,\mathrm{Tr}(\mu M\Sigma)\mathrm{Tr}(\partial_\mu\Sigma\,\partial^\mu\Sigma^\dagger)+\mathrm{H.c.}\\&+c_6(f^2/\Lambda^2)\,\mathrm{Tr}(\mu M\,\partial_\mu\Sigma\,\partial^\mu\Sigma^\dagger)+c_7(f^2/\Lambda^2)\,\mathrm{Tr}(\partial^2\Sigma\,\partial^2\Sigma^\dagger).\end{aligned}\tag{1}$$

In the above expression Λ has dimensions of mass, and, as argued in Ref. 2, the coefficients c_i are ~ 1 if one takes $\Lambda\simeq 4\pi f\simeq 1$ GeV. The parameter μ also has dimensions of mass. Since it always multiplies M, one can never determine the quark masses from the chiral Lagrangean, but only mass ratios.

Since we are only interested in the terms of Eq. (1) which are quadratic in the π fields, we can simplify $\mathcal{L}$ by setting c_6 and c_7 equal to zero with no loss of generality. To do this, note that the transformation

$$\Sigma\rightarrow\Sigma\{1+ia[\Sigma^\dagger\partial^2\Sigma-\tfrac{1}{3}\mathrm{Tr}(\Sigma^\dagger\partial^2\Sigma)+\mathrm{H.c.}]+ib[\mu M\Sigma-\tfrac{1}{3}\mathrm{Tr}(\mu M\Sigma)+\mathrm{H.c.}]\}\tag{2}$$

is an $SU(3)\otimes SU(3)$ transformation up to second order in ∂^2 and μM. By choosing a amd b appropriately we can shift away c_6 and c_7, absorbing their effects in the coefficients c_1–c_4, as well as other terms which do not concern us. Furthermore, we can set $c_5=0$ as well, since that term only redefines f to $O(\pi^2)$, which we will fit to the experimental value for f_π.

Expanding $\mathcal{L}$ to second order in π_a we get

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a-\Lambda^2\{2\,\mathrm{Tr}\hat{M}\boldsymbol{\pi}^2(\epsilon+\epsilon^2A\,\mathrm{Tr}\hat{M})\\&+\epsilon^2B\,\mathrm{Tr}(\hat{M}^2\boldsymbol{\pi}^2+\hat{M}\boldsymbol{\pi}\hat{M}\boldsymbol{\pi})+\epsilon^2C[\mathrm{Tr}(\hat{M}\boldsymbol{\pi})]^2+(\alpha/4\pi)D\Lambda^2\,\mathrm{Tr}(Q[\boldsymbol{\pi},[\boldsymbol{\pi},Q]])\},\end{aligned}\tag{3}$$

where $\hat{M}=M/m_s=\mathrm{diag}(m_u/m_s,m_d/m_s,1)$, $\epsilon=\mu m_s/\Lambda^2$, $A=(c_1/2+c_2)$, $B=2c_3$, $C=4c_2-2c_1$, and $D=c_4$. Note that ϵ is the expansion parameter for $SU(3)\otimes SU(3)$ chiral perturbation theory.

Counting parameters reveals that there are six degrees of freedom for fitting five meson masses. This explains why m_s/m_d may be expressed as a function of m_u/m_d and the physical meson masses, which gives the equation of the curve in Fig. 1. Including logarithmic effects shifts the curve, but introduces no new degrees of freedom.

In evaluating Eq. (3) we have neglected terms of order m_u^2, m_d^2, $m_u m_d$ since they are much smaller than terms with one power of m_s. It is consistent to retain terms of $O(m_u m_s)$ and drop ones of $O(m_s^3)$ even though the latter are numerically larger. This is easy to see by looking at the combinations $m_{\pi^0}^2$, $m_{K^+}^2-m_{K^0}^2$ (ignoring electromagnetic splittings for now). These terms get no contribution of $O(m_s^3)$, by isospin invariance. Thus we can determine them to $O(\epsilon)$. Dropping $m_d m_s^2$ or $m_u m_s^2$ terms is equivalent to ignoring contributions of $O(\epsilon^2)$ to their masses. As for $m_{K^+}^2+m_{K^0}^2$ or m_η^2, keeping m_s^2 terms determines the $O(\epsilon)$ corrections to these masses, while m_s^3 terms give $O(\epsilon^2)$ corrections. Thus neglecting terms of order m_u^2, m_d^2, $m_u m_d$ consistently determines all the quark masses to $O(\epsilon)$. The contributions to these masses from Eq. (3) are given in Table I.

Our approximation of neglecting terms of $O(m_u^2,m_d^2,m_u m_d)$ allows us to ignore the effects of η-π^0 mixing. However, one expects η-η' mixing to shift the square of the η mass by $O(\epsilon)$ and so it must be included. Luckily no new term need be added to Eq.

TABLE I. Contributions to the meson masses from Eq. (3). $x \equiv m_u/m_s$, $y \equiv m_d/m_s$.

	$\Lambda^2(\epsilon+\epsilon^2 A\,\mathrm{Tr}\hat{M})$	$\epsilon^2 B\Lambda^2$	$\epsilon^2 C\Lambda^2$	$(\alpha/4\pi)D\Lambda^4$
$m^2_{\pi^0}$	$x+y$	0	0	0
$m^2_{\pi^\pm}$	$x+y$	0	0	1
$m^2_{K^\pm}$	$x+1$	$\frac{1}{2}(2x+1)$	0	1
$m^2_{K^0,\bar{K}^0}$	$y+1$	$\frac{1}{2}(2y+1)$	0	0
m^2_η	$\frac{1}{3}(x+y+4)$	$\frac{4}{3}$	$\frac{2}{3}(1-x-y)$	0

(2). Note that the coefficient C shifts the lowest-order η mass squared by $O(\epsilon)$ without affecting the other mesons. Thus any effect due to η-η' mixing can be absorbed into the coefficient C.

There are also the nonanalytic chiral logarithms which arise from loop graphs which contribute to both mass and wave-function renormalization. The loop momentum is cut off at a scale $\Lambda \sim 1$ GeV. We keep only the logarithmic pieces of these graphs, because the nonanalytic pieces can be absorbed into a redefinition of the (unknown) parameters c_i.

Including the one-loop effects, we find (in an obvious notation)

$$Zm^2_{\text{physical}} = m^2_{\text{higher order}} + m^2_{\text{one loop}} + m^2_{\text{lowest order}}. \quad (4)$$

The one-loop values for Z and m^2 have been calculated previously.[3] Here we use values which are similar to Ref. 3, except that we have been careful to use different masses for (K^+, K^0) and (π^+, π^0) since we are eventually interested in isospin-nonconserving quantities.

We also performed the one-loop correction to f_π, yielding

$$f_\pi = f\left[1 - \frac{1}{2}\frac{m^2_{K^+}}{(4\pi f)^2}\ln\frac{m^2_{K^+}}{\Lambda^2}\right], \quad (5)$$

allowing us to determine the parameter f in terms of the experimental value $f_\pi = 93$ MeV. f is not needed for evaluation of the analytic mass terms from Eq. (3), but it is necessary for evaluation of the logarithmic contributions to the masses. In all the loop calculations we have consistently dropped terms of $O(m_u^2, m_d^2, m_u m_d)$.

The computation can now be summarized. We have fitted m^2_{physical} using Eq. (4) and the calculated values of Z and $(m^2)_{\text{one loop}}$. There are five masses, and six independent parameters. Thus the allowed region is a curve. We have included only those values where $\epsilon = \mu m_s/\Lambda^2 \leq 0.3$. This is because one expects corrections to K-decay processes to be of $O(\epsilon)$. It is hard to understand how these calculations could work to 25% accuracy if ϵ were very large. The second cut we have made is to restrict the net higher-order contribution to be less than a certain fraction F of m^2_{phys}, separately for each meson. Thus, e.g.,

$$|\Lambda^2\epsilon^2 A\,\mathrm{Tr}\hat{M}(y+1) + \tfrac{1}{2}\epsilon^2 B\Lambda^2(1+2y)| \leq F m^2_{K^+}, \text{ etc.} \quad (6)$$

We chose F to be 0.2, 0.25, and 0.3 for the three allowed regions graphed in Fig. 1.

What about baryons? Though we have not discussed them here, we have also fitted our masses to the baryon sector. Unfortunately, the baryons do not provide any useful constraints on the quark-mass ratios. The reason is that there are ten independent operators which contribute to the eight baryon masses, which easily allow one to fit the masses for all values on the m_u/m_d vs m_s/m_d plot in Fig. 1 which we derived from the meson sector. Thus baryons add no additional constraints.

The most striking feature of our result is that a massless up quark is not in contradiction with the past successes of SU(3) ⊗ SU(3) chiral perturbation theory. A massless up quark is an interesting possibility, as it naturally explains the absence of CP nonconservation in the strong interactions.[4] Anyone who has an acquaintance with the extensive literature on the quark masses will be skeptical that $m_u = 0$ is possible: Numerous papers claim to have shown, by means of lowest-order chiral pertrubation calculations, that $m_u = 0$ is in contradiction with not only the meson and bayron masses,[1] but also with isospin-nonconserving effects such as $\eta \rightarrow 3\pi$, and Σ^0-Λ mixing.[5] The point of our paper, though, is that second-order effects can contribute to an effective up-quark mass, of size

$$m_u^{\text{eff}} \sim c m_d \mu m_s/\Lambda^2 \sim c m_d m_K^2/\Lambda^2 \sim \tfrac{1}{4} c m_d, \quad (7)$$

where $c \sim 1$. Note that for $c \simeq 2$, one gets the usual lowest-order result that $m_u \simeq \frac{1}{2} m_d$.

One might wonder whether m_u^{eff} would be able to mimic the lowest-order effects of a real up-quark mass. Indeed it can; consider the matrix identity[6]

$$(\det M)\mathrm{Tr}(M^{-1}\Sigma) = \tfrac{1}{2}\{[\mathrm{Tr}(\Sigma^\dagger M)]^2 - \mathrm{Tr}[(\Sigma^\dagger M)^2]\}. \quad (8)$$

The right-hand side is a second-order operator included in the Lagrangean of Eq. (1), and the above relation says that such an operator is equivalent to addition of a contribution to the quark mass matrix of the form

$(\det M)M^{-1}$. Note that as $m_u \to 0$, we get

$$(\det M)M^{-1} = \begin{pmatrix} m_d m_s & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad (m_u = 0). \tag{9}$$

Thus the second-order operator in Eq. (8) yields an m_u^{eff} such as in Eq. (7) which enters the chiral Lagrangean to lowest order in *exactly* the same way as a real up-quark mass would.[7] *Thus there is no first-order chiral perturbation calculation which is inconsistent with* $m_u = 0$ *when one includes the effects of the second-order operator in Eq. (8).*

Our analysis actually does better than merely replace m_u by the operator in Eq. (8). We fit the meson masses *exactly.* The other operators which enter do not have the same form as the lowest-order mass term. Thus by looking at isospin-nonconserving processes, such as $\eta \to 3\pi$, it may be possible to place limits on the second-order coefficients.

Regardless of whether or not $m_u = 0$, our results show how Weinberg's results for the quark-mass ratios are modified by second-order effects in chiral-symmetry breaking. Any theory which predicts the quark-mass ratios or relates mixing angles to mass ratios will have to be consistent with the results of Fig. 1.

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[1]S. Weinberg, Trans. N. Y. Acad. Sci. **38**, 185 (1977).

[2]S. Weinberg, Physica (Amsterdam) **96A**, 327 (1979); A. Manohar and H. Georgi, Nucl. Phys. **234B**, 189 (1984).

[3]H. Pagels, Phys. Rep. **16C**, 219 (1975).

[4]G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976), and Phys. Rev. D **14**, 3432 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976); C. Callan, R. Dashen, and D. Gross, Phys. Lett. **63B**, 334 (1976).

[5]See, for example, D. Gross, S. Treiman, and F. Wilczek, Phys. Rev. D **19**, 2188 (1979).

[6]An effective up-quark mass of this form in Eq. (7) was first considered by D. G. Caldi, Phys. Rev. Lett. **39**, 121 (1977), and by H. Georgi and I. McArthur, Harvard University Report No. HUTP-81/A011 (to be published), in connection with instanton effects, which contribute to the operator in Eq. (8). The identity is easily proven by expanding $\det(1 + M\Sigma^\dagger) = \exp(\mathrm{Tr}\ln[1 + M\Sigma^\dagger])$.

[7]There is nothing special about the meson sector: $M^{\text{eff}} = (\det M)M^{-1}$ has the same SU(3) ⊗ SU(3) transformation properties as M, and can appear in the baryon sector as well. Of course, the coefficient of M^{eff} need not be the same in the two sectors.