Resonance due to a Local Spin Rotator in High-Energy Accelerators

S. Y. Lee and S. Tepikian
Brookhaven National Laboratory, Upton, New York 11973
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A method of solving the spin equation in a large circular accelerator with a local spin rotator ("snake") is proposed and compared with a numerical simulation of the spin tracking. We found that (1) the envelope function of the snake structure plays an important role in the spin restoration in passing through the resonance, and (2) a new type of spin depolarization resonance, called snake resonance, appears.

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Preservation of spin polarization of a particle beam to highest energy [e.g., at the Superconductivity Super Collider (SSC)] is an important problem in accelerator physics.[1] The spin of a particle in a circular accelerator will precess about a vertical axis as a result of the guide field B with frequency γG with respect to the azimuthal angle θ, where γ is the Lorentz relativistic factor and $G=(g-2)/2$ is the Pauli anomalous magnetic moment.[2,3]

Besides the vertical field B to guide the particle around the circular accelerator, strong-focusing quadrupoles are needed to confine the beam particles to a small size. The beam particles will then execute betatron motion around the central orbit.[4] This betatron oscillation has a frequency called betatron tune ν, which is defined to be the number of oscillations per revolution. Particles displaced vertically from the central orbit in quadrupoles will experience horizontal fields, which gives rise to small perturbing kicks to the spin orientation of the particle. These perturbing kicks are normally unimportant unless the precession frequency is in phase with the kicks. This is a resonance condition. Resonances arising from the accelerator structure are called intrinsic resonances. The one of concern here will appear when the precession frequency $\gamma G = kP \pm \nu$, where k is an integer and P is the periodicity of the machine. At the resonance, the kicks to the spin orientation add up every turn (or periodic structure). In addition, some perturbing kicks arise from the randomly distributed alignment errors in the magnets. These kicks repeat every turn in the accelerator, and therefore a resonance occurs when the precession frequency becomes an integer. The largest resonance $|\epsilon|$ of most of the accelerators in operation or in the design stage has been compiled recently,[5] where the resonance strength $|\epsilon|$ is the Fourier amplitude of the perturbing kicks. In dealing with a low-energy machine such as the Brookhaven National Laboratory alternating-gradient synchrotron or the CERN proton synchrotron, where $|\epsilon| \lesssim 0.15$, a tune jump and orbit corrector has been used to correct each spin resonance separately.[6] For a higher-energy machine, such as the relativistic heavy-ion collider (RHIC), the CERN Super Proton Synchrotron (SPS), and Tevatron (Fermilab) with $|\epsilon| \lesssim 0.75$ and the SSC with $|\epsilon| \lesssim 8$, the conventional method becomes impractical.

"Snakes" were invented to cure the depolarization resonances.[7] A snake is a combination of horizontal- and vertical-field dipoles to rotate the particle spin by π radians about a horizontal axis locally without perturbing the particle orbits outside the snake. With two snakes, the spin is up in half of the accelerator and down in the other half of the accelerator. The spin tune of the machine is therefore $\frac{1}{2}$. This would allow the acceleration of particles without their passing through resonance.[8-10] A simplistic view is that the perturbing kicks due to half the accelerator are cancelled by the perturbing kicks in the other half. This paper is concerned with the case where these perturbing kicks add rather than cancel.

The spin-transfer matrix for the two-component spinor wave function for polarized particles in the accelerator encountering a single spin-depolarizing resonance can be expressed as[11]

$$\psi(\theta_f) \equiv t(\theta_f,\theta_i)\psi(\theta_i), \quad (1)$$

where θ_i and θ_f are the initial and final angles around the accelerator,

$$t(\theta_f,\theta_i) = aC[\alpha - K(\theta_f-\theta_i)/2] + bS[\beta + \tfrac{1}{2}K(\theta_f+\theta_i)], \quad (2)$$

with

$$C(x) \equiv 1\cos x + i\sigma_3 \sin x,$$

$$S(x) \equiv i\sigma_1 \cos x + i\sigma_2 \sin x,$$

$$b = (|\epsilon|/\lambda)\sin[\tfrac{1}{2}\lambda(\theta_f-\theta_i)] = (1-a^2)^{1/2},$$

$$\alpha = \tan^{-1}\{(\delta/\lambda)\tan[\tfrac{1}{2}\lambda(\theta_f-\theta_i)]\},$$

$$\delta = K - \gamma G,$$

$$\beta = \arg\epsilon^*,$$

$$\lambda = (\gamma^2 + |\epsilon|^2)^{1/2},$$

and where σ_i, K, and ϵ are respectively the Pauli matrices, the resonance position, and the resonance strength, and δ is the distance between the spin precession frequency γG and the depolarizing resonance kick frequency K.

The spin wave function at the snake will be transformed locally according to

$$\psi(\theta_s^+) = e^{i(\pi/2)\hat{\mathbf{n}}_s\cdot\boldsymbol{\sigma}}\psi(\theta_s^-) = S(\phi_s)\Psi(\theta_s^-), \qquad (3)$$

where $\hat{\mathbf{n}}_s = (\cos\phi_s, \sin\phi_s, 0)$ denotes the snake axis with respect to the horizontal outward direction $\hat{\mathbf{x}}$ or $\hat{\sigma}_i$ and θ_s^- and θ_s^+ are angles before and after the snake.

For the simplicity of our discussion, we assumed that there are $N_s/2$ pairs of ϕ_1, ϕ_2 snakes evenly distributed in the accelerator at

$$\theta_0, \theta_0 + 2\pi/N_s, \ldots, \theta_0 + (N_s - 1)2\pi/N_s.$$

The snake configuration can be generalized easily. The spin-transfer matrix through a pair of snakes ϕ_1, ϕ_2 is

$$\begin{aligned}\tau(\theta_0+4\pi/N_s,\theta_0) &= t(\theta_0+4\pi/N_s,\theta_0+2\pi/N_s)S(\phi_2)t(\theta_0+2\pi/N_s,\theta_0)S(\phi_1)\\ &= -C(\phi_1-\phi_2)+2b^2\cos(\phi_2-\beta-K\theta_0-2K\pi/N_s)C(\phi_1-\beta-K\theta_0-2K\pi/N_s)\\ &\quad -2ab\cos(\phi_2-\beta-K\theta_0-2K\pi/N_s)S(\phi_1-\alpha+2K\pi/N_s), \qquad (4)\end{aligned}$$

where the components of τ can be obtained easily from C and S matrices with $\tau_{21} = \tau_{12}^*$ and $\tau_{22} = \tau_{11}^*$. The parameters, a, b, α, β, and δ, varying slowly with respect to θ, are taken to be constant in obtaining Eq. (4). Without loss of generality, we now assume that the spin is tracked through the accelerator with a ϕ_1 and ϕ_2 pair of snakes. The spin-transfer matrix after the nth traversal of the ϕ_1, ϕ_2 pair of snakes is

$$T(\theta_{n+1}) = \tau(\theta_{n+1},\theta_n)T(\theta_n). \qquad (5)$$

We shall propose a linear-response approximation to solve Eq. (5), i.e.,

$$T_{12}(\theta_{n+1}) = -e^{i(\phi_1-\phi_2)}T_{12}(\theta_n) + \tau_{12}(\theta_{n+1},\theta_n)(-)^n e^{-in(\phi_1-\phi_2)}.$$

To first order, the spin-transfer matrix for passage through the pair of snakes n times has the following solution:

$$T_{12}(\theta_n) = i2ab(-)^n\exp\left[i\left(\phi_1+\alpha-\frac{2K\pi}{N_s}+(n-1)(\phi_1-\phi_2)\right)\right]\frac{1}{2}\left(C_-\frac{\sin(n\xi_-)}{\sin\xi_-}+C_+\frac{\sin(n\xi_+)}{\sin\xi_+}\right) \qquad (6)$$

with

$$\begin{aligned}C_\pm &= \exp\{\pm i[\phi_2-\beta-K\theta_0-2K\pi/N_s\mp(n-1)\xi_\mp]\},\\ \xi_\pm &= \phi_1-\phi_2\pm 2K\pi/N_s \equiv (2\pi/N_s)(\nu_s\pm K).\end{aligned} \qquad (7)$$

The polarization of the beam is given by

$$\langle S\rangle = 1-2|T_{12}|^2, \qquad (8)$$

which will fall within the envelope function of

$$\langle\bar{S}\rangle = 1-8a^2b^2. \qquad (9)$$

Figure 1 compares the polarization, obtained from the numerical-simulation program written by Buon[12] and Courant,[13] turn by turn with the envelope function of Eq. (9). The spin-tracking program of Buon and Courant[12,13] uses equally spaced quadrupoles, dipoles, and snakes to simulate the spin kicks in the accelerator. The agreement between the theoretical result of Eqs. (8) and (9) and the numerical result is good. Around the spin resonance, the envelope function has many nodes where $\langle\bar{S}\rangle = 1$ and $ab = 0$. These nodes play an imporant role in the spin restoration after passage through the spin resonance. In other cases, when the polarization is lost because of the snake resonances, the node position remains unchanged (but $\langle\bar{S}\rangle$ may no longer be 1).

When the condition $\xi_\pm = k\pi$, $k =$ integer, is satisfied, the perturbing kicks due to the spin resonance for passage through two snakes add up. This is the first-order snake resonance condition. The driving force for the spin depolarization is

$$\sin(n\xi_\pm)/\sin\xi_\pm. \qquad (10)$$

At the resonance, the depolarization driving force is proportional to n, the number of traversals of snake pairs. Figure 2 shows the final polarization from the spin-tracking simulation for particles passing through a single resonance of an accelerator with two snakes as a function of the fractional betatron tune at resonance strength of 1.93. Besides the first-order resonance at $\xi_\pm = k\pi$, there are higher-order snake resonances. We note that the width of the snake resonance at betatron tune $\frac{1}{2}$ is about 0.1. This resonance width can be seen intuitively from the resonance driving force of

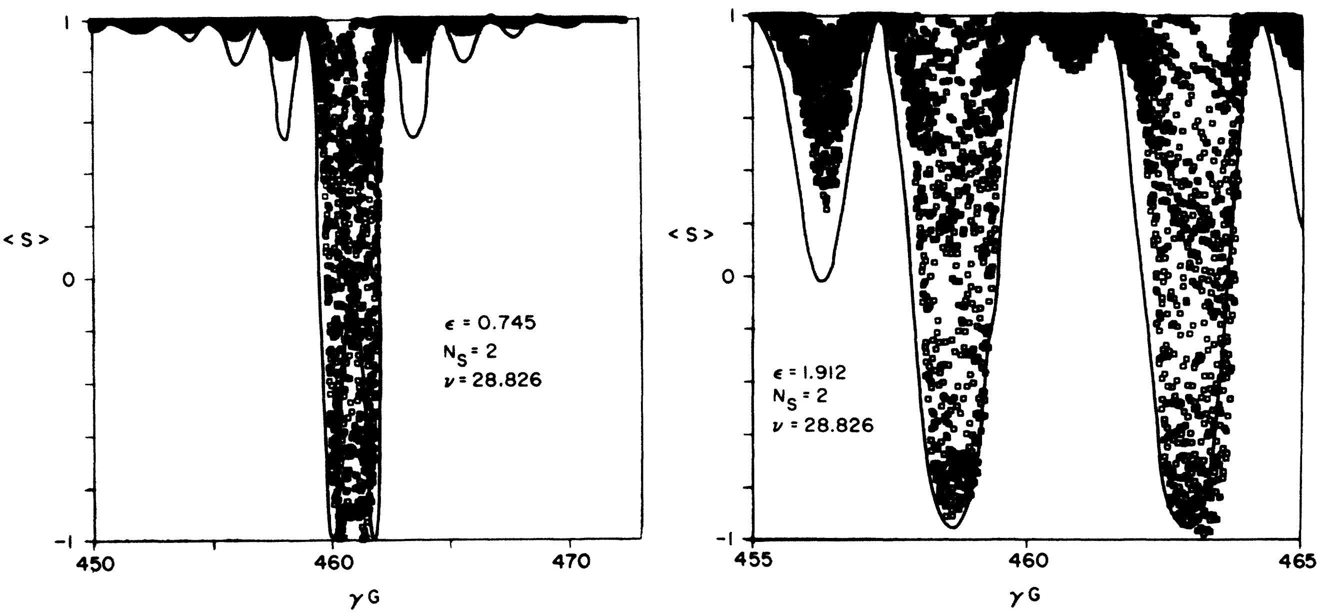

FIG. 1. The polarization at each turn around the accelerator as the particle is accelerated through a spin resonance. The spin-tracking program, written by Buon (Ref. 12) and modified by Courant (Ref. 13), uses equally spaced quadrupoles to simulate the depolarization kick in the accelerator. The envelope function of Eq. (9) is also shown for comparison.

Eq. (10). To understand the higher-order snake resonances, we first consider the exact transfer matrix for the spin passing through two pairs of snakes, i.e.,

$$\tau^{(2)} = \tau\left(\theta_0 + \frac{8\pi}{N_s}, \theta_0 + \frac{4\pi}{N_s}\right)\tau\left(\theta_0 + \frac{4\pi}{N_s}, \theta_0\right). \qquad (11)$$

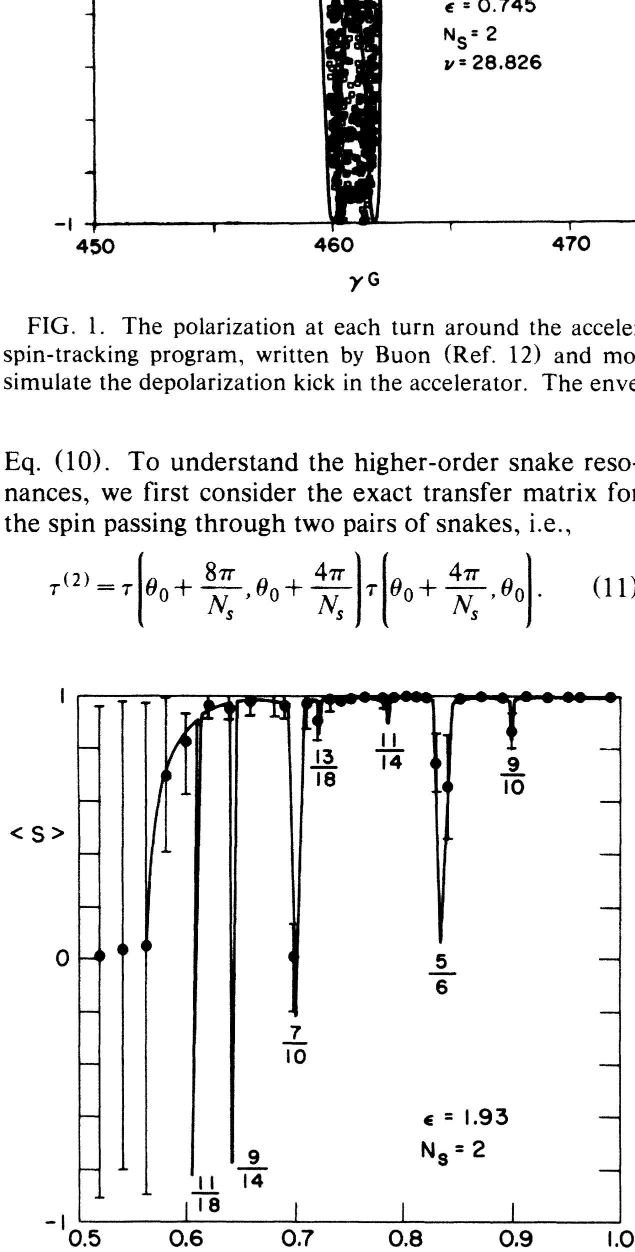

FIG. 2. The polarization of the beam after passage through a single spin resonance at strength $\epsilon = 1.93$ as a function of the fractional part of the spin resonance. Several higher-order resonances are seen easily.

With use of Eq. (11) as the driving term in the spin tracking of Eq. (5) (note that here, $\theta_{i+1} = \theta_i + 8\pi/N_s$), the linear part of the solution reproduces Eq. (6). The mth-order contribution to the depolarizing driving force after nth traversal of the m concatenated snake pairs is

$$ab^{(2m-1)} \sin(n\xi_{\pm}^{(m,l)})/\sin\xi_{\pm}^{(m,l)}, \qquad (12)$$

with the order $m = 2$ and $l = 1$ or 3, and

$$\xi_{\pm}^{(m,l)} = m\frac{2}{N_s}(\nu_s \mp lK)\pi, \qquad (13)$$

$$l = 1, 3, \ldots, (2m-1).$$

When $\xi_{\pm}^{(m,l)}/\pi =$ integer is satisfied, the depolarization driving force of Eq. (12) is proportional to n for traversal through nm pairs of snakes.

In general, the mth-order snake resonance is obtained by concatenation of m pairs of snakes into $\tau^{(m)}(\theta_0 + 4m\pi/N_s, \theta_0)$, and solve Eq. (5) with $\tau^{(m)}$. The snake resonance will appear at the betatron tune, which satisfies $\xi^{(m,l)}/\pi =$ integer of Eq. (12). However, at an even m, the spin tune of the concatenated spin-transfer matrix does not exhibit the snake structure. Thus the even-order snake resonance should not appear. Numerically, two snakes can restore spin with spin resonance at fractional tunes of $\frac{2}{6}$, $\frac{4}{6}$ (second order) and $\frac{5}{8}$, $\frac{6}{8}$ (fourth order) with $|\epsilon| \geq 2.05$. This suggests that the snake resonance of even order is irrelevant.

Table I lists snake resonances up to the third order.

TABLE I. Snake resonance at $\nu_s = \frac{1}{2}$ for two snakes.

Order	Resonances	Fractional part of K							
1	$(\nu_s \pm K)$	$\frac{1}{2}$							
3	$3(\nu_s \pm K)$	$\frac{1}{2}$					$\frac{5}{6}$		
	$3(\nu_s \pm 3K)$	$\frac{1}{2}$		$\frac{11}{18}$		$\frac{13}{18}$	$\frac{5}{6}$	$\frac{17}{18}$	
	$3(\nu_s \pm 5K)$	$\frac{1}{2}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{7}{10}$	$\frac{23}{30}$	$\frac{5}{6}$	$\frac{9}{10}$	$\frac{29}{30}$

The width of mth-order resonance, corresponding to $\xi^{(m,l)}/\pi =$ integer, is approximately $0.1/ml$. Up to the fifth-order resonance, the spacing between successive resonances is about 0.02; therefore the tune spread of the machine must be maintained within the spacing between these resonances. We found that the critical resonance strength ϵ_c for two snakes is about $\epsilon_c \simeq 3$ at an appropriately chosen tune. However, at the third-order resonance (e.g., $\nu = 28 \pm \frac{1}{30}$), the critical resonance strength becomes $\epsilon_c \simeq 0.75$. This indicates that two snakes are sufficient for RHIC, SPS, and Tevatron and the snake resonance up to third order is important.

In conclusion, we have developed a systematic method of analyzing the particle spin in the accelerator with snakes. The theoretical calculation agrees very well with the numerical simulations. Because of the snakes, the polarization of the particles can be restored after passage through the spin resonance provided that the snake resonances are avoided. The snake resonances of order m due to the concatenation of m pairs of snakes at $\xi_{\pm}^{(m,l)}/\pi =$ integer come from the beating between the spin tune due to spin resonance and the snake tune of the snake configuration. These snake resonances are clearly observed in the numerical simulation.

Our method of analysis can be applied directly to the polarized proton beam in SSC, where the resonance strength at the top energies is about $|\epsilon| \simeq 8$. At an appropriate choice of the betatron tune, six to twelve snakes may be needed for polarized protons. However, since the width of the snake resonance is proportional to $N_s/2$, or equivalently the snake superperiodicity P_s, it is preferable to have a lower snake superperiodicity. The snake configuration should be chosen to cancel the imperfection resonance in linear order. Careful analysis of this problem could be addressed in the SSC lattice design for polarized protons.

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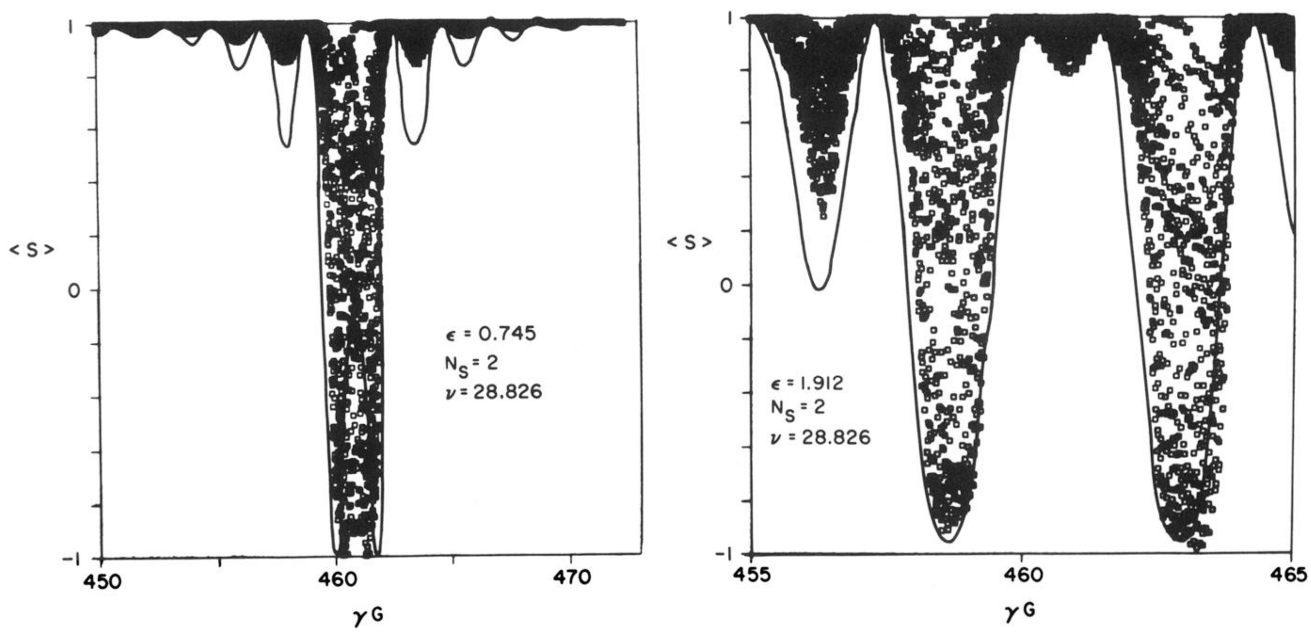

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