Sensitivity of a Hopf Bifurcation to Multiplicative Colored Noise

R. Lefever and J. Wm. Turner
Chimie Physique II, University of Brussels, B-1050 Brussels, Belgium
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The location of a Hopf bifurcation in parameter space may be postponed or advanced by multiplicative colored noise depending on the interplay between the time scale of the noise and the rotation period of the phase variable.

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The effect of multiplicative noise on nonequilibrium systems is currently the object of considerable research, particularly into the capacity of such noise to give rise to so-called "noise-induced transitions."[1] The latter can be viewed, in most experimental examples known so far,[2–6] as a shift of a preexisting deterministic instability: The noise displaces, by an amount proportional to its intensity, the location in parameter space of an instability already present under noiseless conditions.

In the analysis of this behavior it is usual (i) to take as a starting point the Landau equation, or more generally the normal-form equation, giving the long-time deterministic dynamics of some slow mode(s) near the instability[7]; (ii) to take the noise into account by letting some parameter fluctuate in this reduced evolution equation; and (iii) to consider that the noise is white. This is in line with (i): The noise indeed is not affected by the instability; hence, the closer one is to the instability point, the more separated the time scales of the slow mode(s) and of the noise become and the more appropriate the white-noise idealization appears.

The reliability of these simplifications, however, does not depend solely on the occurrence of a critical slowing down; it also requires that the coupling of the noise with the "fast" variables, which do not slow down at the instability, be appropriately modeled. This condition has been scarcely studied. Usually the role of the fast variables is overlooked[8] and the adequacy of (i)–(iii) is simply postulated; the justification comes *a posteriori* insofar as the outcome of the analysis agrees with the experimental results. This is, however, not always the case; e.g., the noise-induced shifts of electrohydrodynamic instabilities in liquid-crystal systems cannot be explained by the addition of white noise to the bifurcation parameter of the Landau equation for the unstable mode.[6,9,10]

Our motivation here is to investigate the coupling between noise and a fast variable in the case of a well-known chemical model (Brusselator)[11] presenting an instability frequently met in nonequilibrium systems: the Hopf bifurcation. We find that even in the limit of vanishingly small noise intensity, it is the interplay between the noise and the "fast" phase variable which determines whether the bifurcation is postponed or advanced.

The effect of multiplicative white noise has been considered recently in diverse physicochemical oscillators.[12] The approaches based on normal forms[13–15] predict that the noise tends to stabilize the trivial state.[16] Furthermore if, as recommended in Ref. 1, the bifurcation of the most probable value of the probability density is taken as index of bifurcation, one finds that the noise postpones the instability by an amount proportional to its intensity. The reason is easy to see on the normal-form equations of a Hopf bifurcation: $\dot{\rho} = \lambda\rho - \mathrm{Re}K\rho^3$ and $\theta = \Omega - \mathrm{Im}K\rho^2$. ρ describes the radial motion undergoing a critical slowing down when the bifurcation parameter $\lambda \to 0$. θ is the phase, Ω the rotation period, and K a parameter. If white-noise fluctuating forces are superimposed on the constant average value of λ and Ω, it is clear (since the equation for ρ is decoupled from the one of θ) that ρ is a stochastic process behaving essentially like x in the Landau equation $\dot{x} = \lambda x - x^3$ in which a white noise of intensity σ^2 is superimposed on λ. As is well known,[1,2] the bifurcation of the extremum of the probability density for x is then postponed by the noise from $\lambda = 0$ (deterministic case) to the value $\lambda = \sigma^2/2$. For ρ, this implies that the limit-cycle onset is postponed.

We now consider the Brusselator model:

$$\dot{X} = A - X + X^2 Y - BX, \quad \dot{Y} = BX - X^2 Y. \tag{1}$$

X and Y are reactant concentrations; A and B are control parameters. The unique steady-state solution $X_s = A$, $Y_s = B/A$ undergoes a Hopf bifurcation at $B = B_c = 1 + A^2$. We study the effect of the fluctuations of B on this transition when the average $\langle B \rangle$ is constant and close to B_c: We put $\langle B \rangle - B_c = l\epsilon^2$, where ϵ is a small parameter and $l = 1, 0, -1$ for $\langle B \rangle$ above, at, or below B_c. We do not, however, invoke the proximity of the bifurcation point to adopt the white-noise idealization [cf. (iii)]. More realistically, we model the noise by a colored Ornstein-Uhlenbek process z_t obeying the stochastic differential equation

(SDE)

$$dz_t = -\gamma K^{-2} z_t\,dt + K^{-1}\sigma\,dW_t. \qquad (2)$$

W_t is the Wiener process. The scaling parameter K appearing in (2) provides a convenient measure of the "distance" from the white-noise situation[17] (see also Ref. 1, Chap. 8): For $K \to 0$, the correlation time of the noise, $\tau_c = K^2/\gamma$, goes to zero; the stationary probability density of z, given by

$$p_s(z) = (2\pi\bar{\sigma}^2)^{-1/2}\exp[-(z^2/\bar{\sigma}^2)/2],$$

with $\bar{\sigma}^2 = \sigma^2/2\gamma$, however, is independent of K. Three time scales govern the dynamics: $\tau_m = 1/\epsilon^2$ which describes the radial relaxation in the X-Y phase space, $T = 1/A$ which is the linear period of rotation, and τ_c which is the correlation time of the noise. Hence, the system's properties can be unfolded in terms of two dimensionless parameters:

$$\mu \equiv (T/\tau_m)^{1/2} = \epsilon/A^{1/2},$$

$$1/\eta \equiv (T/\tau_c)^{1/2} = \gamma^{1/2}/(A^{1/2}K).$$

We replace B by $B(z_t) = \langle B\rangle + \mu^\alpha \eta^{-\beta} z_t$ in Eqs. (1) and (2) (α and β are exponents to be fixed later according to the situation investigated), set $\tau = At$, and introduce polar coordinates through the transformation

$$X = A + \mu A^{1/2}u^{1/2}\cos\theta,$$

$$Y = A^{-1} + A + I\mu^2 + \mu A^{1/2}u^{1/2}(\sin\theta - \cos\theta).$$

The Fokker-Planck equation (FPE) corresponding to the SDE's for u, θ, and z resulting from these transformations and giving the evolution of the joint probability density $p(u,\theta,z)$ reads

$$\begin{aligned}[p(u,\theta z)]^{-1}\partial_\tau p(u,\theta,z) = &-\partial_u\{\mu[2(A^{-3/2} - A^{1/2})c^3u^{3/2} + 4A^{-1/2}c^2su^{3/2}]\\ &+ 2\mu^2(c^2Iu - c^4u^2 + A^{-1}c^3su^2) + 2\mu^3(A^{-1/2}Ic^3u^{3/2})\\ &- 2A^{-1/2}\mu^{\alpha-1}\eta^{-\beta}z(cu^{1/2} + A^{-1/2}\mu c^2u)\}\\ &-\partial_\theta\{-1 + \mu[(A^{1/2} - A^{-3/2})c^2su^{1/2} - 2A^{-1/2}cs^2u^{1/2}]\\ &+ \mu^2(c^3su - csI - A^{-1}c^2s^2u) - \mu^3(A^{-1/2}c^2sIu^{1/2})\\ &+ A^{-1/2}\mu^{\alpha-1}\eta^{-\beta}z(su^{-1/2} + \mu A^{-1/2}cs)\} + \eta^{-2}(\partial_z z + \bar{\sigma}^2\partial_z z); \qquad (3)\end{aligned}$$

$c = \cos\theta$ and $s = \sin\theta$. We are interested in the stationary solution $p_s(u,\theta,z)$ of (3). Since B is close to B_c, we admit that the inequalities $\mu \ll 1$ and $\mu \ll 1/\eta$ which express the slowing down of the radial relaxation are fulfilled. The objective is then to determine the position of the bifurcation point for the two cases where the phase is either faster or slower than the noise. We take the bifurcation of the extremum of the reduced stationary probability density

$$p^*(u) = \int_R\int_0^{2\pi} dz\,d\theta\,p_s(u,\theta,z)$$

as indicator of the transition.

(1) $\eta^{-2} \ll 1$ *or, equivalently,* $T = A^{-1} \ll \tau_c = \gamma^{-1}K^2$. *The phase evolves more rapidly than the noise.*—It is appropriate to modulate the intensity of the noise in (3) by putting $\alpha = 1$ and $\beta = 0$. Indeed, from the correlation function $\langle z_\tau z_0\rangle = (\bar{\sigma}^2/2)\times\exp(-|\tau|/\eta^2)$ we note that η^2 is nothing else than the correlation time of the noise when the unit of time is chosen equal to the rotation period. Hence, in the limit $\eta^{-2} \to 0$, which is of interest here, the evolution of θ is completely correlated to that of z: The spectral density $S(\nu)$ of z_τ converges to a δ peak located at the frequency $\nu = 0$, i.e., $\lim_{\eta^2\to 0} S(\nu) = (\bar{\sigma}^2/2)\delta(\nu)$. This is exactly what one expects to happen when the evolution of θ is so fast that at every instant the value of this variable is able to "equilibrate" to the fluctuations of the noise (for more details see Arnold, Horsthemke, and Lefever,[18] and Ref. 1, p. 226). By use of this scaling and choice of A so that the mathematical condition $\tau_c \ll A^{-1/2}$ ($A \ll 1$) is satisfied, the stationary probability density $p_s(u,\theta,z)$ can be expanded as

$$p_s(u,\theta,z) = p_0(u,\theta,z) + \eta^{-2}p_2(u,\theta,z) + \eta^{-4}p_4(u,\theta,z) + \ldots.$$

Replacing in (3), one easily finds that $p_0(u,\theta,z) = \phi(u,z)/2\pi$, where $\phi(u,z)$ is the solution of

$$\{\partial_u[u^2(Iu - 3u^2/4) - \mu A^{-1}zu] - (\partial_z z + \bar{\sigma}\,\partial_z z)\}\phi(u,z) = 0.$$

Expanding now $\phi(u,z)$ as

$$\phi(u,z) = \phi_0(u,z) + \mu\phi_1(u,z) + \mu^2\phi_2(u,z)\ \ldots,$$

one finds that $\phi_0(u,z)=p_s(z)p^*(u)$, where $p_s(z)$ is the stationary solution of the FPE for the Ornstein-Uhlenbeck process and $p^*(u)$ is the solution of

$$[-\overline{\sigma}^2A^{-2}\partial_u u\,\partial_u u+\partial_u(Iu-3u^2/4)]p^*(u)=0.$$

This equation is familiar: By a simple redefinition of its parameters it transforms into the steady-state version of the FPE associated with the well-known Verhulst SDE: $dx_t=\lambda x-x^2+\sigma x\,dW_t$. The latter is, with the Landau equation, a standard paradigm for the study of noise-induced shifts. Indeed for both systems, the stationary probability density of x undergoes an abrupt transition at $\lambda=\sigma^2/2$: The extremum which for $\lambda<\sigma^2/2$ is at $x=0$ switches abruptly to a nonzero value of x for $\lambda>\sigma^2/2$. For the Hopf bifurcation considered here, this behavior corresponds to a postponement of the oscillatory regime (limit cycle) towards a higher value of the control parameter $\langle B\rangle$. This is in agreement with the result obtained by the addition of Gaussian white noise to the bifurcation parameter in the normal-form equation (cf. above). In this shortcut procedure, however, the condition that the phase must be much more rapid than the noise is only implicit. That it is essential can be seen from the case considered now.

(2) $\mu \ll 1 \ll \eta^{-1}$. *The phase evolves more slowly than the noise.*—The noise being now the fastest process in the system, we modulate its intensity in (3) so that for $\eta^{-1}\to\infty$, the white-noise limit is recovered. This requires that $\beta=1$. Indeed, it is easy to verify that when $\eta\to 0$, the correlation function of z_τ/η, i.e.,

$$\langle(z_\tau/\eta)(z_0/\eta)\rangle=(\overline{\sigma}^2/2\eta^2)\exp(-|\tau|/\eta^2),$$

becomes δ correlated and that the corresponding spectral density, i.e., $S(\nu)=\overline{\sigma}^2/[2\pi(\eta^4\nu^2+1)]$, converges for all frequencies ν towards the constant value $\overline{\sigma}^2/2\pi$; in other words, the power spectrum becomes white. Furthermore, we shall investigate the system's behavior in the weak-noise limit obtained by our putting $\alpha=2$. The stationary solution of (3) can then be expanded in powers of η. At the lowest order one finds that $p_0(u,\theta,z)=p_s(z)p_s(u,\theta)$. Indeed, in the limit $\eta\to 0$, the noise becomes δ correlated.[1] The joint probability density $p_s(u,\theta)$ is the stationary solution of the FPE (without loss of generality we may take here that $A=1$):

$$\begin{aligned}
[p_s(u,\theta)]^{-1}\partial_\theta p_s(u,\theta)=&\ \mu[4c^2s\,\partial_u u^{3/2}-\partial_\theta(2cs^2u^{1/2})]\\
&+\mu^2\{c^2[\partial_u(2Iu+\overline{\sigma}^2)-2\overline{\sigma}^2\partial_{uu}u]-2c^4\partial_u u^2+2c^3s\,\partial_u u^2+\overline{\sigma}^2s^2\partial_u\\
&\quad+\partial_\theta[cs(\overline{\sigma}^2u^{-1}-I+2\overline{\sigma}^2\partial_u)+c^3su-c^2s^2u-\partial_\theta(\overline{\sigma}^2s^2u^{-1}/2)]\}\\
&+\mu^3\{c^3\partial_u(2Iu^{3/2}+3\overline{\sigma}^2u^{1/2}-4\overline{\sigma}^2\partial_u u^{3/2})+3\overline{\sigma}^2cs^2\partial_u u^{1/2}\\
&\quad+\partial_\theta[c^2s(3\overline{\sigma}^2u^{-1/2}/2-Iu^{1/2}+4\overline{\sigma}^2\partial_u u^{1/2})-\overline{\sigma}^2s^3u^{-1/2}/2-\overline{\sigma}^2\partial_\theta(cs^2u^{-1/2})]\}\\
&+\mu^4\overline{\sigma}^2\{2(c^4\partial_u u-\partial_{uu}u^2+c^2s^2\partial_u u)+\partial_\theta[c^3s(\tfrac{1}{2}+2\partial_u u)-cs^3/2-\partial_\theta(c^2s^2/2)]\}.
\end{aligned}$$

Expanding $p_s(u,\theta)$ as

$$p_s(u,\theta)=p_0(u,\theta)+\mu p_1(u,\theta)+\mu^2p_2(u,\theta)+\ldots,$$

one finds that up to the order μ^2

$$\begin{aligned}
p_s(u,\theta)=&\ (N/2\pi)\{1-\mu[c^3(4u^{3/2}/3)\partial_u+2cu^{1/2}]\\
&+\mu^2[C+65u^3/(576\overline{\sigma}^2)-Iu^2/(4\overline{\sigma}^2)-25u/48+c^6(8u^3/9)\partial_{uu}\\
&\quad+c^4(25u^2/6)\partial_u+3c^2u-c^3s(u^2/2)\partial_u+cs(\overline{\sigma}^2\partial_u-I)]\}\exp[(Iu-3u^2/8)/\overline{\sigma}^2],
\end{aligned}$$

where N is a normalization constant and C is a constant to be determined from the normalization condition $\int_0\int_0^{2\pi}du\,d\theta\,p_2(u,\theta)=0$. For $\overline{\sigma}^2\ll 1$, one has $C\simeq 43/243\overline{\sigma}^2+\frac{10}{27}$. Calculating the reduced probability density $p^*(u)$ for the action variable u, one easily finds from the extremal condition $\partial_u p^*(u)=0$ that the extremum u_m which corresponds to the amplitude of the limit cycle is now located at

$$u_m=\tfrac{4}{3}+\mu^2(-\tfrac{352}{81}+47\overline{\sigma}^2/36).$$

Clearly the amplitude of the limit cycle as given by u_m is larger in the presence of noise than without. This shows that in contrast with case (1), the bifurcation point is advanced compared to its position under deterministic conditions.

These results show that the response to multiplicative noise of systems undergoing a pitchfork bifurcation, like, e.g., the chemical oscillator considered here, the single-mode laser near threshold, the Couette and Bénard instabilities, or various electrical circuits, is not as universal as the studies based on reduced dynamical descriptions predict. Whether or not, in a given system, the noise postpones such a bifurcation depends not only on the coupling of the noise with the slow mode(s), but also, and in an essential manner, on its coupling with the fast variables. The outcome of the latter effect, especially when the noise is the fastest process in the system as under the conditions of case (2) above, can in general not be accounted for by the dynamics valid locally near a bifurcation point. The interplay between the noise and dynamical features specific to each model system in particular becomes essential.

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