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# Inequality for the Infinite-Cluster Density in Bernoulli Percolation

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Under a certain assumption (which is satisfied whenever there is a dense infinite cluster in the half-space), we prove a differential inequality for the infinite-cluster density, $P_\infty(p)$, in Bernoulli percolation. The principal implication of this result is that if $P_\infty(p)$ vanishes with critical exponent $\beta$, then $\beta$ obeys the mean-field bound $\beta \leq 1$. As a corollary, we also derive an inequality relating the backbone density, the truncated susceptibility, and the infinite-cluster density.

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The percolation model was introduced in 1957 by Broadbent and Hammersley[1] to describe the diffusion of fluid through a porous medium. Because of its simplicity and its often surprising applicability, percolation has become one of the most widely studied and best understood models in statistical physics. Although the theoretical framework of percolation is by now well established,[2] many questions in the rigorous analysis of percolation theory remain open.[3] The purpose of this Letter is to address one of these questions.

The Bernoulli site-percolation model is defined as follows[4]: Each site of $Z^d$ is taken to be occupied with an independent, homogeneous probability $p$, and vacant with probability $1-p$. Perhaps the most fundamental quantity in percolation theory is the infinite-cluster density, $P_\infty(p)$, which is the probability that a given site (say, the origin) is in an infinite cluster of occupied sites. The infinite-cluster density is the conventional order parameter for percolation; hence, the percolation threshold, $p_c$, is defined to be the smallest value of $p$ above which $P_\infty > 0$.

It is generally accepted[2] that $P_\infty(p)$ exhibits scaling behavior as $p \to p_c^{\,+}$:

$$P_\infty(p) \sim |p-p_c|^\beta. \tag{1}$$

The only lattice for which such behavior is rigorously known to occur is the Cayley tree, on which one finds a relation of the form (1) with $\beta=1$. It should be noted that the tree value $\beta=1$ distinguishes the mean-field behavior of percolation from that of the Gaussian (or Ising-type) models.[5]

In this Letter, we prove that under a certain assumption there is a *lower bound* of the form (1) with $\beta=1$ in all dimensions. We thus derive the mean-field bound for the infinite-cluster density exponent:

$$\beta \leq 1. \tag{2}$$

It is expected[2] that the bound saturates for $d \geq d_c = 6$. Our result complements much recent work[6-8] on the rigorous analysis of critical exponents for percolation theory.

The basis of our mean-field bound is a new differential inequality for the infinite-cluster density:

$$P_\infty \leq \frac{P_\infty}{p}\frac{d(pP_\infty)}{dp} = p^{-1}P_\infty^2 + P_\infty \frac{dP_\infty}{dp}. \tag{3}$$

We prove (3) under a relatively weak assumption (explained below) that is known to hold for all $p$ in $d=2$ and expected to hold for all $p$ in all dimensions. The proof of (3) is surprisingly easy; indeed, after preliminary definitions and review of some results in rigorous percolation theory, the essentials of the proof are but a few lines and will be presented in this Letter. First, however, let us show that (3) implies the asserted mean-field bound.

Take $p > p_c$, which implies $P_\infty(p) > 0$, and integrate (3) from $p_c$ to $p$. With the fact that $p > p_c$ and expansion of the resulting expression, this gives

$$P_\infty(p) - P_\infty(p_c) \geq [1 - p_c^{-1}P_\infty(p_c)](p-p_c) + O((p-p_c)^2), \tag{4}$$

which implies that $P_\infty(p)$ does not tend to its limiting value slower than linearly. Indeed, assuming that $P_\infty(p_c)=0$ (which is rigorously known[9] in $d=2$ and expected in all $d$), (4) reduces to

$$P_\infty(p) \geq (p-p_c)+O((p-p_c)^2). \qquad (5)$$

Thus if $P_\infty(p)$ has scaling behavior as in (1), the exponent $\beta$ obeys the mean-field bound $\beta \leq 1$.

Most of the rest of this Letter is devoted to the proof of the differential inequality (3). However, at the end of the Letter, we will also show that reasoning along the lines of this proof can be used to derive a new inequality relating the so-called "backbone" density, the truncated susceptibility, and the infinite-cluster density.

*Definitions and statement of the hypothesis.*—Our inequality is derived by our viewing an infinite cluster as being composed of a "backbone" and "dangling ends." The backbone of an infinite cluster has been described in various contexts as the part of the infinite cluster which carries current. Thus, in the infinite-volume limit, a natural definition is that a site is in the backbone if it is occupied and attached to two disjoint infinite paths of occupied sites. We define the backbone density, $Q_\infty(p)$, to be the probability that any given site (say, the origin) is in the backbone. For convenience, we also define $\mathbf{P}_\infty(s)$ and $\mathbf{Q}_\infty(s)$ to be the events that the site $s$ is part of an infinite cluster or part of a backbone, respectively, so that $P_\infty(p) = \mathrm{Prob}[\mathbf{P}_\infty(0)]$ and $Q_\infty(p)=\mathrm{Prob}[\mathbf{Q}_\infty(0)]$.

It is possible to envisage infinite clusters with no backbone. We will call these "spineless infinite cluster," and define $S_\infty(p)$ to be the event that the origin is part of a spineless infinite cluster. Note that $S_\infty(p)=0$ trivially for $p<p_c$. Although it is intuitively obvious that spineless infinite clusters should, with probability 1, also not appear when $p \geq p_c$, this result can only be rigorously demonstrated in $d=2$, where it follows from the work of Harris.[10] For $p$ above threshold in $d \geq 3$, a proof that $S_\infty(p)=0$ requires certain additional assumptions, the weakest of which (at present) is that one can find a dense infinite cluster in the half-space.

The hypothesis under which we prove our inequality (3) is that $S_\infty(p)=0$. If this is the case, then whenever the origin is in an infinite cluster, it is (with probability 1) either part of a backbone or part of a dangling end. The latter event is then simply $\mathbf{P}_\infty(0)\backslash\mathbf{Q}_\infty(0)$, where the backslash denotes deletion. Thus,

$$P_\infty(p)=Q_\infty(p)+\mathrm{Prob}[\mathbf{P}_\infty(0)\backslash\mathbf{Q}_\infty(0)]. \qquad (6)$$

The point of our proof is to bound both terms on the right-hand side of (6) by functions of the infinite-cluster density. For this we need the following.

*Some results of rigorous percolation theory.*—First, we define *positive* and *negative* events. Let $\omega$ be a configuration of occupied and vacant sites. An event $A$ is said to be positive if it is the case that whenever $A$ occurs in a given configuration $\omega$, it also occurs in all configurations $\omega'$ with the property that each site which is occupied in $\omega$ is also occupied in $\omega'$. Negative events are defined analogously with "occupied" replaced by "vacant." Note that the events $\mathbf{P}_\infty(0)$ and $\mathbf{Q}_\infty(0)$ are both positive, while the event that the origin is in a dangling end is neither positive nor negative.

Next, we define the notion of *articulation sites.* A site $s$ is said to be an articulation site for an event $A$ if alteration of the configuration at the site $s$ changes the status of the event $A$. We denote by $\delta_s A$ the event that $s$ is an articulation site for the event $A$. Note that $\mathbf{Q}_\infty(0)$ is precisely the event that $\mathbf{P}_\infty(0)$ occurs and that there are no articulation sites (other than the origin) for $\mathbf{P}_\infty(0)$. Articulation sites are useful in the context of the following formula, derived by Russo[9]: If $A$ is a local positive event, then

$$d(\mathrm{Prob}[A])/dp=\textstyle\sum_s \mathrm{Prob}[\delta_s A]. \qquad (7)$$

Furthermore (7) also applies to certain nonlocal positive events, including the event $\mathbf{P}_\infty(0)$ (see, e.g., Durrett[11]). Notice that the right-hand side of (7) is just the expected number of articulation sites for the event $A$. Special cases of this formula have also appeared in the work of Coniglio.[12]

Finally, we describe the notion of *events which occur disjointly.* Let $U \subset Z^d$ and let $A$ be an event. Then $A_{|U}$ is notation for the event that $A$ occurs and cannot be destroyed by altering any sites in the complement of $U$; for obvious reasons, we say that $A_{|U}$ is the event that "$A$ occurs on the set $U$." Two events $A$ and $B$ are said to occur disjointly, denoted by $A \circ B$, if there are disjoint sets, $U$, $V \subset Z^d$, $U \cap V = \phi$, such that both $A_{|U}$ and $B_{|V}$ occur. For the case in which both $A$ and $B$ are positive events, van den Berg and Kesten[7] have proved the inequality

$$\mathrm{Prob}[A \circ B] \leq \mathrm{Prob}[A]\mathrm{Prob}[B]. \qquad (8)$$

With use of (8), it is not hard to derive that

$$Q_\infty(p) \leq p^{-1}P_\infty^2(p), \qquad (9)$$

which itself has some interesting consequences for the random-resistor-network model.[13] Indeed, (9) follows simply from (8) and the observation that $\mathbf{Q}_\infty(0)$ is the disjoint occurrence of the event $\mathbf{P}_\infty(0)$ and the event that a neighbor of the origin is part of an infinite cluster.

A recent improvement of (8), due to van den Berg and Fiebig,[14] extends the inequality to the case in which $A$ and $B$ are (each) intersections of a positive and a negative event.[15] This extension considerably simplifies our derivation of (3).

*Proof of the differential inequality (3).*—We begin

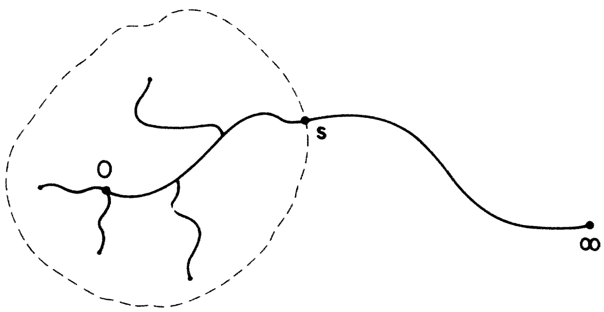


FIG. 1. The event $\delta_s \mathbf{P}_\infty(0)$.

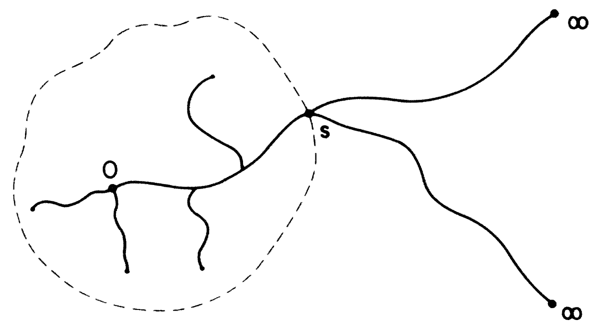


FIG. 2. The event $\mathbf{D}_s(0)$.

with the identity (6). By (9), we see that we already have (the easy) half of the proof. Thus, it suffices to estimate the "dangling end" term $\text{Prob}[\mathbf{P}_\infty(0)\backslash\mathbf{Q}_\infty(0)]$. To this end, consider the event $\delta_s\mathbf{P}_\infty(0)$. By definition, whenever $\delta_s\mathbf{P}_\infty(0)$ occurs and $s$ is occupied, removal of $s$ disconnects the origin from infinity. Dually, this means that, if $s$ is vacant, there is a "cutting surface" (of vacant sites) surrounding the origin and passing through $s$, and that *all* cutting surfaces surrounding the origin pass through $s$. (That is, the "other side" of $s$ is connected to infinity.) This may be expressed graphically as in Fig. 1. It is evident that $\delta_s\mathbf{P}_\infty(0)$ is the intersection of a negative and a positive event.

Now, let us discuss the dangling-end event $\mathbf{P}_\infty(0)\backslash\mathbf{Q}_\infty(0)$. If this occurs, and $S_\infty(p)=0$, we claim that there is a unique site $s\in Z^d$ which is (i) part of the backbone of the infinite cluster, and (ii) an articulation site for the event $\mathbf{P}_\infty(0)$. To see this, one first needs the easily derivable fact that if $s'$ and $s''$ are both part of the backbone of an infinite cluster ($s'\neq s''$), then there are two mutually disjoint infinite paths of occupied sites, one containing $s'$ and the other containing $s''$. Now denote by $C_{\text{DE}}(0)$ the set of occupied sites in the dangling end of the origin [i.e., $C_{\text{DE}}(0)$ contains those sites connected to the origin by a path of occupied sites, none of which are in the backbone]. Clearly, there is at least one site $s$ in the backbone with a neighbor in $C_{\text{DE}}(0)$. To show that $s$ is unique, one simply assumes the opposite and uses the above (easily derivable) fact to show that this implies the existence of an $x\in C_{\text{DE}}(0)$ which is also in the backbone—a contradiction. That this unique $s$ in the backbone is also an articulation site for $\mathbf{P}_\infty(0)$ follows from a similar line of reasoning.

Let us partition the event that the origin is in a dangling end according to which site $s$ satisfies (i) and (ii) above. This gives

$$\mathbf{P}_\infty(0)\backslash\mathbf{Q}_\infty(0)=\cup_s\mathbf{D}_s(0), \tag{10}$$

where $\mathbf{D}_s(0)$ is the event that $s$ satisfies (i) and (ii). Graphically, $\mathbf{D}_s(0)$ may be represented as in Fig. 2, from which it is obvious that

$$\mathbf{D}_s(0)=\delta_s\mathbf{P}_\infty(0)\circ\mathbf{P}_\infty(s). \tag{11}$$

Now, using the inequality (8) (as extended by van den Berg and Fiebig[14]) and the Russo[9] formula (7), we have

$$\text{Prob}[\mathbf{P}_\infty(0)\backslash\mathbf{Q}_\infty(0)]\leq P_\infty(p)d[P_\infty(p)]/dp, \tag{12}$$

which completes the proof of (3).

*An inequality for the backbone density.*—Reasoning analogous to the preceding proof also shows that $\text{Prob}[\mathbf{D}_s(0)]$ is bounded above by $Q_\infty(p)\tau'_{0\partial s}$, where $\tau'_{0\partial s}$ is the probability that origin and a neighbor of $s$ are in the same finite cluster. Summing over $s$, one obtains

$$P_\infty(p)\leq Q_\infty(p)[1+2d\chi'(p)], \tag{13}$$

where $2d$ is the coordination number of the lattice, and $\chi'(p)$ is the expected size of finite clusters, also identified as the (truncated) susceptibility. Assuming (1) and that $Q_\infty(p)\sim|p-p_c|^q$, $\chi'(p)\sim|p-p_c|^{\gamma'}$, we have

$$q\leq\gamma'+\beta \tag{14}$$

which supplements the lower bound[8] $q\geq 2\beta$, derivative from (9). It is worth noting that these upper and lower bounds agree in mean field.

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[4]Percolation can, of course, be defined for bonds rather than sites, and for lattices other than $Z^d$. Our principal results hold for these cases also, although the terms in the inequality (3) are modified by (nonsingular) multiplicative factors.

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[13]Equation (9) was originally derived for the bond model in Ref. 8. The site version stated here differs by a factor of $p^2$.

[14]J. van den Berg and U. Fiebig, "On a Combinatorial Conjecture Concerning Disjoint Occurrence of Events," Ann. Probab. (to be published).

[15]Of course, a positive event is a degenerate case of the intersection of a positive and a negative event.