Ordered Phase of Short-Range Ising Spin-Glasses

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We propose a new picture of the Ising-spin-glass phase, based on an *Ansatz* for the scaling of low-lying large-scale-droplet excitations. We find behavior very different from the infinite-range model. The truncated spatial correlations decay as a power of distance, the ac nonlinear susceptibility diverges as a power of $\ln\omega$, and the magnetization noise power diverges as $1/\omega$ with logarithmic corrections. A magnetic field destroys the spin-glass phase so that there is no de Almeida–Thouless transition. Defect excitations should yield similar dynamic phenomena in vector spin-glasses.

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There has recently been considerable progress in understanding the static and dynamic properties of the ordered phase of the infinite-range Ising spin-glass [the Sherrington-Kirkpatrick (SK) model].[1,2] However, there has been very little success either in extending the SK results away from the infinite-range limit or in directly attacking the short-range models of interest for any spatial dimensionality, d. Computer simulations[3] and numerical approximate renormalization calculations[4] on the $d=3$ Edwards-Anderson (EA) model do indicate, however, a finite-temperature ordering transition.

In this paper we propose a new picture of the ordered spin-glass phase in short-range systems and discuss static and dynamic properties based on a simple scaling *Ansatz* motivated by the results of the numerical "domain-wall" renormalization-group studies of Bray and Moore, and McMillan.[4] We assume that there is an ordered phase with nonzero EA order parameter[5] $q_{\rm EA}=\langle\langle S_i\rangle_t^2\rangle_c$ for temperatures $T<T_c$. The brackets $\langle\ldots\rangle_c$ denote the configurational average and the brackets $\langle\ldots\rangle_t$ an infinite-time average in the phase of interest.[5]

The low-lying excitations which dominate the long-distance and long-time correlations in the Ising-spin-glass phase are clusters or droplets of coherently flipped spins (see below). Our basic assumption is that the density of states at zero energy for such droplets at length scale L scales as $L^{-\theta}$, with $0<\theta\leqslant(d-1)/2$. At large L the thermally excited droplets are therefore dilute and may be treated as noninteracting two-level systems.[6,7] With the further assumption that the activation barriers scale as L^{ψ}, we show that the autocorrelation function

$$C(t)=\langle\langle S_i(0)S_i(t)\rangle_t-\langle S_i\rangle_t^2\rangle_c \tag{1}$$

decays as $(\ln t)^{-\theta/\psi}$ for $t\to\infty$. This results in $1/f$ noise for the magnetization, as has been observed in some insulating spin-glasses.[8] We also find that the EA correlation function falls off at long distances as $r^{-\theta}$ and that the ac nonlinear susceptibility diverges as a power of $\ln\omega$ as frequency $\omega\to0$.

The picture of the ordered phase which emerges is in striking contrast to that of the SK model. From a renormalization-group point of view, the results that we obtain are consequences of static and dynamic scaling which arise from the approach to a disordered zero-temperature fixed point with eigenvalue $-\theta$. The scaling behavior is quite analogous to that at the random-field Ising critical point analyzed recently.[9]

We first consider the $T=0$ behavior of the $S_i=\pm1$ Ising spin-glass with a continuous distribution of exchanges $\{J_{ij}\}$. In a ferromagnet or in the unfrustrated Mattis model[10] of a spin-glass, the stiffness of the ordered phase (or ground state) can be obtained by measurement of the average difference in energy for systems of linear dimensions L between periodic and antiperiodic boundary conditions in one of the directions. This domain-wall energy is σL^{d-1} with σ the interfacial tension. In a true spin-glass, because of the frustration present on all length scales and the resultant indeterminacy in which boundary conditions are more favorable, the situation is quite different. The authors of Ref. 4 have calculated the difference in ground-state energies for samples of Ising spin-glass of size L^d between periodic and antiperiodic boundary conditions. The higher-energy configuration can be thought of as an imposed domain wall on one side of which the spins are reversed from the other lower-energy state. It is natural to expect that for large L, the characteristic magnitude of the energy difference is ΥL^{θ}, with $\theta<d-1$ and Υ the appropriate generalization of the twist modulus or interfacial tension. The results[4] are consistent with this form with $\theta\simeq0.2$ for $d=3$. (When $\theta<0$ the spin-glass phase does not exist for $T>0$; this appears to occur for $d=2$.[4]) The distribution of domain-wall energies is found to be broad with weight at zero.[4] This should be expected for a frustrated spin-glass since, as argued below, the domain-wall energy is subadditive.

For a symmetric distribution of exchanges the pairs of samples of size L^d which differ only in having periodic and antiperiodic boundary conditions are drawn from an ensemble with bonds differing only along one surface of area L^{d-1}. If the dependence of the ground-state energy on the exchanges had only short-range correlations, the rms energy variations within this ensemble would scale as $L^{(d-1)/2}$, yielding $\theta \leq (d-1)/2$. Although there are, in fact, long-range correlations in the effects of altered bonds, an argument to be presented elsewhere shows that this naive bound for θ is nevertheless correct.

We hypothesize the following picture for the low-lying excitations from a ground state. By definition the energy of such a state cannot be lowered by the flipping of any *finite* collection of spins. We expect domain-wall or droplet excitations on a scale L which consist of closed surfaces surrounding of order L^d spins flipped from the ground state. We consider only the lowest-energy such droplet in each region of volume L^d. The boundaries of these low-energy droplets are presumably fractal with a surface area of order L^{d_s} with $d-1 \leq d_s \leq d$ but for $\theta > 0$ the droplets should be compact, with $d_s < d$. We first make a simple model for the statistics of these droplets which, as we will later argue, should correctly yield the essential features at long distances and times for all $T < T_c$ after appropriate coarse graining has been performed to take into account interactions between droplets. For $T > 0$ we must consider free energies instead of energies.

Consider an ensemble of independent (but overlapping) droplets. At each size L^d they are, from the definition above, distributed with spatial density $\delta L/L^{(d+1)}$ for droplets of size between L^d and $(L+\delta L)^d$. Thus each spin is, on average, in of order one droplet of size between L^d and $(2L)^d$. Motivated by the results of Ref. 4 we take the probability distribution for the excitation free energies F_L of droplets on scale L to have the scaling form

$$P_L(F_L) \approx \rho(F_L/\Upsilon L^\theta)/\Upsilon L^\theta \tag{2}$$

for large L, with $\rho(0) > 0$.

The behavior of the ordered phase is dominated by the thermally active droplets: those with free energy less than or of order of T. As long as θ is positive only a fraction of order $T/\Upsilon L^\theta$ of the droplets are active for L large, and only a fraction of order T of the spins lie in active droplets. Thus there is a linear low-T specific heat and the static susceptibility is nonzero even at $T=0$. These are simply consequences of a nonzero density of states of two-level systems at zero energy[6] and are dominated by the small active droplets. The long-distance behavior of the correlation function, $C_{ij} = \langle S_i S_j \rangle_t - \langle S_i \rangle_t \langle S_j \rangle_t$, on the other hand, is dominated by the large droplets. For spin separation $|i-j|$ large the dominant contributions to C_{ij} come from droplets which include *both* i and j.[11] These give rise to exponentially small correlations unless a droplet containing both sites is active in which case C_{ij} has magnitude of order q_{EA}. Averaging thus yields $\langle C_{ij}^2 \rangle_c \sim q_{\mathrm{EA}}^2 T/(\Upsilon |i-j|^\theta)$. The slow falloff for $d=3$ may explain why the system appears critical for $T < T_c$.[3]

The nonlinear effects of a weak magnetic field H can be considered by a generalization of the Imry-Ma[12] argument. A droplet has magnetization of order $L^{d/2}$ so that any field aligns the large droplets since $\theta < d/2$. This implies that the long-range order is destroyed by nonzero uniform (or random) field.[13] The magnetization, m, induced by a small field has a singular part[14] which scales as $m \sim H^{d/(d-2\theta)}$ since droplets of scale $L \gtrsim H^{-2/(d-2\theta)}$ independently align with the field. For $0 < T < T_c$, the EA susceptibility, $\chi_{\mathrm{EA}} = \sum_{ij} \langle C_{ij}^2 \rangle_c$ diverges (since $\theta < d$). The nonlinear susceptibility $\chi_{\mathrm{nl}} = \partial^3 m/\partial H^3$ diverges less strongly because of cancellations: Its divergence is related to the *deviation* of $\rho(\epsilon)$ from $\rho(0)$ for small ϵ. If $\rho(\epsilon) - \rho(0) \sim \epsilon^\phi$ then χ_{nl} is infinite if $d > (1+\phi)\theta$, which is likely to be the case for $d=3$.

We now turn to the low-frequency dynamics of the spin-glass phase, which, as for disordered ferromagnets,[15] is dominated by large-droplet excitations. In order to proceed, we need to know the time scales of the droplet fluctuations. To form (grow) a droplet of scale L, it is necessary to go over a free-energy barrier, B, which is typically larger than the excitation free energy of the completed droplet since the domain wall must pass through regions of high free-energy cost while the droplet grows. We make the natural conjecture that the typical barrier scales as $B \approx \Delta L^\psi$ with $\theta \leq \psi \leq d-1$. A droplet with barrier B lasts for a time t of order $t_0 e^{B/T}$, where t_0 is a microscopic time. As long as the distribution of B is well behaved (i.e., without a long power-law tail), we can obtain the correct long-time form of dynamic correlation functions by considering only typical barriers.

The autocorrelation function, $C(t)$, Eq. (1), is dominated at long times by the active droplets with barriers of order $T\ln(t/t_0)$. It follows that

$$C(t) \sim \frac{q_{\mathrm{EA}} T}{\Upsilon} \left[\frac{\Delta}{T\ln(t/t_0)} \right]^{\theta/\psi}, \tag{3}$$

i.e., there is an extremely slow logarithmic decay of temporal correlations. The magnetization correlations decay in the same way as long as there is only short-range ferromagnetic or antiferromagnetic order. The frequency-dependent magnetization noise spectrum is therefore $C(\omega) \sim 1/\omega |\ln\omega|^{(1+\theta/\psi)}$, yielding $1/f$ noise up to logarithmic corrections, in agreement with recent measurements.[8] The imaginary part of the ac susceptibility is $\chi''(\omega) \sim |\ln\omega|^{-(1+\theta/\psi)}$, and the real part scales as $\chi(0) - \chi'(\omega) \sim |\ln\omega|^{-\theta/\psi}$. An ac nonlinear susceptibility is the cubic 3ω response of the magnetization to a field with frequency ω, which scales as

$$\chi_{\mathrm{nl}}(3\omega;\omega) \sim |\ln\omega|^{[d-(1+\phi)\theta]/\psi}$$

and thus diverges for $\omega \to 0$ when $d > (1+\phi)\theta$.

Having stated the main predictions in terms of noninteracting droplets, we now discuss the justification for applying the results to short-range spin-glasses at all temperatures where there is a nonzero q_{EA}. It is apparent that it is only the active droplets at any given temperature which play an important role, so that we must define these droplets and analyze the effects of interaction with other droplets on their behavior. Fortunately, as long as the spin-glass ordered fixed point is stable so that $\theta > 0$, the density of large active droplets is small. Consider a large droplet, D, of scale L. For $T > 0$, we coarse grain to a scale $l << L$ taking into account all fluctuating droplets on scales smaller than l which affect the free energy, F_D, of D. This process produces contributions to F_D from fluctuations near the boundary of D which should scale as $L^{d_s/2}$ (the square root of its area) and are thus likely to be larger than $F_D \sim \Upsilon L^\theta$. Thus we must minimize the position and shape of the droplet *after* coarse graining to the scale l. This implies that which large droplets are active varies substantially with temperature since $\theta < d_s/2$, because the droplet free energy ΥL^θ is a near cancellation of much larger energy and entropy terms.[16] More importantly, it also implies that the *relative sign of spins separated by sufficiently long distances changes randomly with any change in temperature* because the minimum-free-energy orientation of large droplets varies. However, at a *fixed* T, it is only important that the large active droplets are well defined; surprisingly in light of the above, this appears to be the case.

We estimate the contribution to the entropy of D from interaction with droplets (i.e., deformations) of scales between l and L. This is bounded by $(L/l)^d$ times the entropy of configurational changes on scale l^d in a region of size of order l^d, which is on average $T/\Upsilon l^\theta$. Thus as long as $(L/l)^d T/\Upsilon l^\theta << 1$, F_D is not appreciably affected by this entropy and the droplet wall, when it is there, spends most of its time in the configuration obtained by minimization of the free energy coarse grained at scale l. This condition is satisfied with high probability for large L provided that $l \sim L^x$ with $d/(d+\theta) < x < 1$. Thus on a scale $l(L)$ active droplets of scale L are well-defined excitations which, if the scaling *Ansatz* for the ground-state excitations are correct, have a density of order $T/\Upsilon L^\theta$.

The coefficients Υ and Δ for the droplet and barrier energies are of course temperature dependent. They are of order $(\langle J_{ij}^2 \rangle_c)^{1/2}$ for $T << T_c$ and by scaling vanish for $T \to T_c$ as $\xi^{-\theta}$ and $\xi^{-\psi}$, respectively, where ξ is the correlation length which diverges at T_c.

Our predictions for the behavior of the short-range Ising spin-glass are in striking contrast to those for the SK model. Most significantly we find that autocorrelations decay as universal powers of $\ln t$, while in the SK model they decay as nonuniversal powers of time.[2] Both systems share the feature, however, of a divergent EA susceptibility in the spin-glass phase. In a magnetic field the SK model still has a phase transition (the de Almeida–Thouless line),[17] while *there is no such transition in the short-range system.* Several authors have attempted to extract the low-temperature behavior of short-range spin-glasses by expanding about the Parisi solution of the SK model.[18] If our picture is correct, this approach must fail. With regard to the distribution $P(q)$ of overlaps between states,[1] for a finite large system of size L^d, $P(q)$ has weight away from the peaks (e.g., at $q=0$) of order $T/\Upsilon q_{EA} L^\theta$ [for T near T_c, $P(q=0)$ increases for $L \lesssim \xi$ before beginning to decay]. However, in the thermodynamic limit $P(0)$ vanishes for short-range spin-glasses with $\theta > 0$, in contrast to the Parisi solution for the SK model.[1,19] In addition, the hierarchy of states and the much touted ultrametricity found in the SK model[1] does not exist here: Each spin is likely to be in either one or, more often, no larger active droplet. In the presence of random power-law interactions with $\langle J_{ij}^2 \rangle_c \sim |i-j|^{-2\sigma}$ a simple argument shows that the present picture is not altered provided that $\sigma > d - \theta$. Thus it should hold for Ruderman-Kittel-Kasuya-Yosida interactions ($\sigma = 3$) in $d=3$. For $d/2 < \sigma \leq d - \theta$, where θ is the exponent with only short-range interactions, the exponent is likely to be $\theta_\sigma = d - \sigma$. Thus it appears that the ordered phase of any physically reasonable spin-glass is qualitatively different from that of the SK model.

So far, we have only discussed fluctuations in equilibrium, which is, of course, extremely difficult to attain below T_c. The results should apply, however, for nonequilibrium states which have relaxed for a time τ provided that the time scale of the measurement is $<< \tau$. This is consistent with the behavior of the magnetization noise found in Ref. 8.

Finally, we note that many of the ideas presented here should apply to vector spin-glasses. As discussed by Henley,[20] vector spin-glasses have droplet defects, involving improper rotations as well as other defects that involve only proper rotations. These should have barriers and low-energy densities of states varying as powers of their length scale, just as for Ising spin-glasses, and will thus dominate the long-time dynamics, since gapless excitations yield correlations decaying as powers of time.[21] For vector spin-glasses the stiffness to improper rotations might scale with a larger exponent θ_i than that for proper rotations, θ_p. For a $d=3$ nearest-neighbor Heisenberg spin-glass a preliminary calculation[22] indicates $\theta_i < 0$ and thus no spin-glass phase. However, for power-law interactions with $\sigma \leq d - \theta_i$ it appears that $\theta_i = \theta_p = d - \sigma$. Thus Ruderman-Kittel-Kasuya-Yosida interactions in $d=3$ may well be marginal. Villain[23] has pointed out that vector spin-glasses have Ising-type multispin local operators that are invariant under proper rotations, but change sign under improper rotations. He suggests[23] that these Ising-type operators order for any d where the Ising spin-glass orders; however, the effective interactions between these operators are not entirely random and are of very long range,[23] which raises doubts about this suggestion.

In conclusion, we have introduced a new consistent picture of the ordered phase of short-range Ising spin-glasses based on a numerically supported[4] scaling *Ansatz.* The static and dynamic correlation functions at long distances and long times are dominated by low-energy large-droplet excitations. Our picture of the ordered phase is very different from that of the SK model and we hope that it will serve as a basis for future investigations of short-range spin-glasses.

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Noted added.—After this paper was submitted, a preprint was received from A. Bovier and J. Fröhlich which independently suggests the existence of low-energy excitations which lead to power-law decay of the truncated correlations.

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