Basis-Independent Tests of *CP* Nonconservation in Fermion-Mass Matrices

M. Gronau and A. Kfir
Physics Department, Technion, Haifa, Israel

and

R. Loewy
Department of Mathematics, Technion, Haifa, Israel
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We derive a set of invariant quantities in fermion-mass matrices, independent of one's weak-eigenstate basis, the vanishing of which is both necessary and sufficient for *CP* invariance. Our method is applied to the standard single-Higgs-doublet SU(2) ⊗ U(1) model with an arbitrary number of fermion generations.

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The origin of *CP* nonconservation observed in the K-$\bar{K}$ system remains one of the problems unresolved by the standard model of the electroweak interactions. In the minimal SU(2) ⊗ U(1) single-Higgs-doublet model, *CP* nonconservation requires complex Yukawa couplings and at least three generations.[1] Indeed, as experiments of the last decade show, nature provides us with at least three families of fermions. The actual overall number of generations, N, is another aspect for which the present theory has no answer. The theoretical possibility of further generations beyond the observed three has recently been the subject of quite extensive study,[2] with special attention to the role played by the new fermions in *CP* nonconservation.

In the standard model the arbitrary $N \times N$ complex fermion-mass matrices, which arise after spontaneous breaking of the gauge symmetry, represent the fermion-mass eigenvalues, the mixing-angle parameters, and the *CP* phases. All these physical quantities may in principle be obtained from the fundamental weak-eigenstate representation of the fermion-mass matrices by a straightforward diagonalization procedure. In such a procedure, complex mass matrices may, however, lead to vanishing *CP* phases. Furthermore, it is not always easy to see which of the phases appearing in the generalized Kobayashi-Maskawa (KM) matrix can be rotated away.[3] An interesting question is, therefore, what are conditions that the fundamental fermion-mass matrices satisfy to violate *CP* invariance? Namely, are there are some invariant quantities in fermion-mass matrices, independent of one's weak-eigenstate basis, the vanishing of which is both necessary and sufficient for *CP* invariance?

Recently, Jarlskog[4] identified the determinant of the commutator of the Hermitian up- and down-quark mass matrices as the corresponding invariant quantity for three generations. A large number of generations requires different and a larger number of conditions. Do such conditions exist and what is their nature? The answer to this question may shed some new light on the relation between the fermion-mass problem and *CP* nonconservation. A first attempt to answer this question was made by Bernabeu, Branco, and Gronau.[5] However, the set of conditions derived from *CP* invariance was not sufficient to guarantee the symmetry. The difficulties in finding sufficient conditions were briefly explained. The purpose of the present note is to show how one may obtain these conditions systematically for any number of generations.

To formulate the question, we consider the relevant terms of the standard-model Lagrangean, after spontaneous symmetry breaking:

$$\mathscr{L} = (\bar{u}\bar{d})_L \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}_R + g\bar{u}_L \gamma_\mu d_L W^\mu + \text{H.c.} + \ldots . \quad (1)$$

The left- and right-handed up (u) and down (d) fields represent N-component weak-eigenstate fields; M_u and M_d are N-dimensional quark-mass matrices. The weak-eigenstate fields are defined up to a common unitary transformation (U_L) applied to both u_L and d_L and up to separate unitary transformations (U_R^u and U_R^d) applied to u_R and d_R. Therefore, *CP* symmetry may be formulated as the symmetry under the set of transformations

$$u_L \to U_L C u_L^*, \quad d_L \to U_L C d_L^*,$$
$$u_R \to U_R^u C u_R^*, \quad d_R \to U_R^d C d_R^*, \quad (2)$$

where C is the Dirac charge-conjugation matrix and U_L, U_R^u, U_R^d are arbitrary unitary matrices. *CP* symmetry of the Lagrangean requires

$$U_L^\dagger M_u U_R^u = M_u^*, \quad U_L^\dagger M_d U_R^d = M_d^*. \quad (3)$$

Namely, it is necessary and sufficient for *CP* invariance that three unitary matrices U_L, U_R^u, U_R^d exist such that Eq. (3) holds. This in turn implies that the two

Hermitian matrices $H_q = M_q M_q^\dagger$ ($q = u,d$) satisfy

$$U_L^\dagger H_u U_L = H_u^*, \quad U_L^\dagger H_d U_L = H_d^*, \tag{4}$$

and vice versa: Eqs. (3) follow from Eqs. (4). This may be shown by an explicit construction of the matrices U_R^q which satisfy Eq. (3).

The existence of a unitary (symmetric) matrix U_L which satisfies both Eqs. (4) is therefore necessary and sufficient for CP invariance.[5] Our purpose here is to find equivalent conditions written in terms of invariants of the Hermitian mass matrices H_u, H_d alone. It turns out that this algebraic problem was treated in a more general context.[6] For completeness we will present here a detailed solution in the special case of interest. For simplicity, albeit not strictly required, it will be assumed that nondegeneracy and nonvanishing of masses holds separately for the up and down quarks. Furthermore, before we proceed we note that the existence of U_L which obeys Eqs. (4) in one weak-eigenstate basis guarantees the existence of such a matrix in any other basis. For convenience, we will first study the question in the special basis in which one of the two matrices, say H_d, is diagonal. In this basis we will determine some general properties of the other matrix, H_u, needed and being sufficient for satisfying Eq. (4). We will then express these properties as basis-independent conditions on H_u and H_d.

To simplify notations, let us denote the diagonal matrix H_d by D and the other matrix H_u by H. In this basis we wish to prove that a matrix U_L satisfying Eqs. (4) exists if, and only if, all the cyclic products of elements of H are real:

$$\operatorname{Im}(H_{i_1 i_2} H_{i_2 i_3} H_{i_3 i_4} \cdots H_{i_n i_1}) = 0. \tag{5}$$

Here and from now on, repeated indices are not summed over unless explicitly specified. Equation (5) applies to any n distinct indices $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, N$ and to $n = 3, 4, \ldots, N$. Since for a Hermitian matrix all cyclic products of orders 1 and 2 are real, this equation starts to be nontrivial for cyclic products of order 3. Therefore, at least three generations are needed for CP nonconservation.[1]

It is simple to see that Eqs. (5) follow from Eqs. (4). Since H_d is diagonal and nondegenerate, to satisfy the first of Eqs. (4) U_L must be a diagonal matrix of phases,

$$(U_L)_{jk} = \delta_{jk} e^{i\alpha_j}. \tag{6}$$

The second of Eqs. (4) then implies that the elements H_{ij} either vanish or have phases given by $\frac{1}{2}(\alpha_i - \alpha_j)$. Hence follows Eq. (5). To prove that the latter is also sufficient for satisfying (4) we will assume with no lack of generality that H is irreducible. If it were reducible, our proof would apply separately to each of its submatrices. An irreducible matrix has the property[7] that for every pair i,f there is at least one sequence of nonvanishing matrix elements which starts at i and ends at f:

$$H_{ij} H_{jk} \cdots H_{rf} \neq 0. \tag{7}$$

To construct the unitary phase matrix U_L of Eq. (6), we arbitrarily choose $\alpha_1 = 0$ and define all the other phases by $(1 < f \leq N)$

$$e^{i\alpha_f} = H_{1i}^* H_{ij}^* \cdots H_{rf}^* / H_{1i} H_{ij} \cdots H_{rf}. \tag{8}$$

Irreducibility implies that at least one such nonvanishing sequence of matrix elements exists. It follows simply from Eq. (5) that any other nonvanishing sequence would lead to the same phase, and hence Eq. (8) is unambiguous. This definition implies that all the nonvanishing elements H_{fg} have phases given by $\frac{1}{2}(\alpha_f - \alpha_g)$, and therefore the matrix U_L of Eq. (6) satisfies Eq. (4).

Our next task is to express Eq. (5) in a basis-independent manner. We will first write down equivalent relations in a rather general manner and then focus on a smaller set of such conditions. We start by observing that Eqs. (4) lead to the following general identity for any integer n:

$$\operatorname{Im}\operatorname{Tr}[P_1(H_d) P_1'(H_u) P_2(H_d) \cdots P_n'(H_u)] = 0, \tag{9}$$

where P_s, P_s' ($s = 1, \ldots, n$) are arbitrary polynomials with real coefficients. Moreover, we will turn the argument around and show that Eq. (9) leads to Eq. (5), and hence is also sufficient to yield Eq. (4). Considering Eq. (9) in the basis in which $H_d = D$, we note that an arbitrary polynomial P_s of this nondegenerate matrix represents a real diagonal matrix with arbitrary elements p_j^s, $P_s(D)_{jk} = \delta_{jk} p_j^s$. When we take $P_s'(H) = H$, Eq. (9) reads

$$\sum_{i,j,k,\ldots,r=1}^{N} \operatorname{Im}(H_{ij} H_{jk} \cdots H_{ri})(p_i^1 p_j^2 \cdots p_r^n) = 0. \tag{10}$$

Since p_j^s are arbitrary, each of the coefficients $\operatorname{Im}(H_{ij} H_{jk} \cdots H_{ri})$ must vanish, so that Eq. (5) is obtained.

The basis-independent conditions, for arbitrary polynomials P_s,

$$\operatorname{Im}\operatorname{Tr}(P_1(H_d) H_u P_2(H_d) H_u \cdots P_n(H_d) H_u) = 0 \quad (3 \leq n \leq N), \tag{11}$$

are therefore necessary and sufficient for CP invariance. Hermiticity of H_u allows our restricting n to the above values. Our next task is to replace these equations by a finite number of explicit conditions on H_u and H_d. This is done in a straightforward manner by use of the linear structure of Eq. (11) in each of the polynomials and the Cayley-Hamilton theorem. These imply that an equivalent set of conditions may be written by replacement of each of the polynomials in Eq. (11) by a single power,

$$P_s(H_d) = I, H_d, H_d^2, \ldots, H_d^{N-1} \quad (1 \le s \le n). \quad (12)$$

This constitutes a set of conditions not all of which are independent. The number of these conditions is of order N^N for large N.

Our final goal is to extract from these equations a smaller set from which all the other relations follow. This will be achieved by our first considering Eqs. (11) and (12) for $n=3$ and then augmenting this set by some independent conditions for $n>3$. Starting with $n=3$, we note that the integer powers of H_d may be taken in increasing order,

$$\operatorname{Im}\operatorname{Tr}(H_d^a H_u H_d^b H_u H_d^c H_u) = 0; \qquad (13)$$
$$0 \le a < b < c \le N-1.$$

This follows from the Hermiticity of H_q and from the cyclic invariance of the trace. The trace for two equal powers ($a=b$, for instance) is always real, and conditions for powers of decreasing order may be reordered into the form of Eq. (13). These equations imply that in the basis in which H_d is diagonal, all the cyclic products of order 3 of elements of H_u are real.

For three generations this is sufficient for CP invariance. Therefore for $N=3$ the single condition[8]

$$\operatorname{Im}\operatorname{Tr}(H_u^2 H_d H_u H_d^2) = 0 \qquad (14)$$

is necessary and sufficient for CP symmetry to hold.

In general one has $\frac{1}{6}N(N-1)(N-2)$ equations of type (13). Whereas they imply that all cyclic products of order 3 in H are real, they do not always lead to real cyclic products which involve a larger number of matrix elements, as required by CP invariance. For the latter to be real, in general, one must also use some of Eqs. (11) and (12) with $3 < n \le N$. We note, however, that if H does not have zero off-diagonal matrix elements then no extra conditions beyond Eqs. (13) are needed. This may be easily seen from identities such as

$$\begin{aligned} H_{ijkl} &\equiv H_{ij} H_{jk} H_{kl} H_{li} \\ &= |H_{ik}|^{-2} (H_{ij} H_{jk} H_{ki})(H_{ik} H_{kl} H_{li}), \\ &= |H_{jl}|^{-2} (H_{li} H_{ij} H_{jl})(H_{lj} H_{jk} H_{kl}), \end{aligned} \qquad (15)$$

and similar identities for cyclic products of higher order.

For the general case of possibly vanishing off-diagonal matrix elements some of the conditions with $n>3$ are needed to yield real cyclic products of order $n>3$. These may be judiciously chosen. We will illustrate a possible choice of these conditions for the two cases $N=4$, $N=5$.

For $N=4$, one must only study the special case of two vanishing off-diagonal matrix elements $H_{ik} = H_{jl} = 0$ where i, j, k, and l are four distinct indices. This is the only case in which reality of the four-cycle H_{ijkl} does not follow from reality of all the three-cycles. To yield a real value for H_{ijkl}, one may use two of Eqs. (11) and (12) with $n=4$. It is straightforward to show that for a nondegenerate H_d a suitable choice of two such conditions is

$$\begin{aligned} &P_1 = P_2 = I, \quad P_3 = H_d, \quad P_4 = H_d^2; \\ &P_1 = P_2 = I, \quad P_3 = H_d, \quad P_4 = H_d^3. \end{aligned} \qquad (16)$$

We conclude that for four generations one has six reality conditions, for the traces of $H_u^2 H_d H_u H_d^2$, $H_u^2 \times H_d H_u H_d^3$, $H_u^2 H_d^2 H_u H_d^3$, $H_u H_d H_u H_d^2 H_u H_d^3$, $H_u^3 H_d H_u \times H_d^2$, $H_u^3 H_d H_u H_d^3$, which are necessary and sufficient for CP invariance.

In a five-generation model, one has ten distinct three-cycle products, the reality of which follows from the ten Eqs. (13). In order for one of the fifteen distinct four-cycles H_{ijkl} not to be necessarily real as a consequence of these equations, the two matrix elements H_{ik}, H_{jl} must vanish. This leads to the vanishing of ten of the four-cycles. It is easy to see that at most three four-cycles may then be nonreal, namely when four off-diagonal elements vanish. These may be, for instance, H_{1234}, H_{1235}, and H_{1435}, when $H_{13} = H_{24} = H_{25} = H_{45} = 0$. It is straightforward to show that the following choice of three of Eqs. (11) with $n=4$ is sufficient for the reality of these three four-cycles:

$$\begin{aligned} &P_1 = P_2 = I, \quad P_3 = H_d, \quad P_4 = H_d^2, \\ &P_1 = P_4 = H_d, \quad P_2 = I, \quad P_3 = H_d^2, \\ &P_1 = P_4 = H_d^2, \quad P_2 = I, \quad P_3 = H_d. \end{aligned} \qquad (17)$$

With Eq. (13), these equations lead to real four-cycles in any other case as well. Finally, there are twelve distinct five-cycles, each made up of the product of five off-diagonal elements. For a five-cycle not to be necessarily real once the three- and four-cycles are real, the remaining five off-diagonal elements must vanish. This leads to the vanishing of all the other five-cycles. It is easy to see that for this special case one may use a single equation to yield a real five-cycle:

$$\operatorname{Tr}[H_u, H_d]^5 = 0. \qquad (18)$$

This condition may be replaced by at most four of Eqs. (11) with $n=5$. Altogether, for $N=5$ the set of necessary and sufficient conditions for CP invariance consists of reality conditions for fourteen traces: ten of type (13) ($n=3$), three applied to the matrices of Eq. (17) ($n=4$), and Eq. (18) ($n=5$).

The two examples of $N=4$ and $N=5$ illustrate the procedure of deriving a set of conditions for *CP* symmetry in the general case of N generations. To conclude this study a few remarks are in order:

(a) The number of basis-independent conditions obtained for *CP* invariance is in general (except in the case $N=3$) larger than $\frac{1}{2}(N-1)(N-2)$, the number of phases in the KM quark-mixing matrix. Moreover, it is also larger than $\frac{1}{6}N(N-1)(N-2)$ the number of distinct three-cycle products in a Hermitian mass matrix. The three-gneration model is a very special case, in which the number of *CP* mass-matrix invariants is equal to the number of KM phases.

(b) The above-derived invariants vanish if, and only if, *CP* is conserved. Still, it is not at all obvious how to use these quantities as measures of *CP* nonconservation, as was suggested in Ref. 4 for the three-generation model. This illustrates again the difficulties of using the quark-mass matrices to define the recently discussed concept of "maximal *CP* nonconservation" in a model-independent manner.[9]

(c) Our method of testing the *CP* invariance of a gauge theory may be applied to other theories. Here the search for invariants of the fundamental fermion-mass matrices must be correlated with a study of the *CP* properties of other sectors which are potential sources of *CP* nonconservation. This may be useful for model building. For leptons this study may be generalized to the case of Dirac and Majorana mass terms. This work is currently in progress.

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