Infinite Conservation Laws in the One-Dimensional Hubbard Model

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We show that the Hamiltonian of the 1D Hubbard model commutes with a one-parameter family of transfer matrices of a new 2D classical model corresponding to two coupled six-vertex models. Central to this result is a new local algebraic relation, a generalization of the (infinitesimal) star-triangle relation.

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The 1D Hubbard model is a tantalizing problem in the field of exactly integrable systems. Lieb and Wu[1] showed that the Bethe-Yang *Ansatz* provides a consistent description of the eigenfunctions of the model.[2] The algebraic structure of the model is, however, not well understood. Typical problems in 1D many-body theory solvable by the Bethe *Ansatz*, such as the *XXZ* model, have a very rich algebraic structure as well. Fundamental to this structure is the fact that the Hamiltonian commutes with a one-parameter family of transfer matrices of an appropriate 2D classical statistical mechanical system.[3] Such a relation implies that the Hamiltonian commutes with an infinite number of "currents," and an analogy with exactly integrable systems in classical mechanics begins to emerge. If different transfer matrices (each commuting with the Hamiltonian) themselves commute,[4] then all the "currents" commute mutually, and the analogy is complete. Relations of the above nature are very important because they not only provide a deeper understanding of the models, but also lead to computationally powerful techniques, such as the algebraic Bethe *Ansatz*.[5]

The present study was motivated by the fact that no nontrivial commuting operators are known for the 1D Hubbard model. We present here an infinite number of currents which commute with the Hamiltonian. This is achieved by the identification of a new model in 2D statistical mechanics, for which the transfer matrix commutes with the Hamiltonian. At the root of the commutation relation is a key result of this work, a local algebraic structure [Eq. (14)], which is a nontrivial generalization of that found by Sutherland for the *XYZ* model.[3,6] The latter is an infinitesimal version of the ubiquitous star-triangle (Yang-Baxter) relation. The star-triangle relation is well known to be a sufficient condition for the integrability of a model, but not necessary. The structure described here is interesting in that it is a new and broader sufficiency condition, at the infinitesimal level.

The Hamiltonian of the 1D Hubbard model is written in the form

$$H = -\sum(C^\dagger_{m+1\sigma}C_{m\sigma} + \mathrm{H.c.}) + U\sum(n_{m\uparrow}-\tfrac{1}{2})(n_{m\downarrow}-\tfrac{1}{2}). \tag{1}$$

By using a Jordan-Wigner transformation,[2] we obtain

$$H = \sum(\sigma^+_{m+1}\sigma^-_m + \mathrm{H.c.}) + \sum(\tau^+_{m+1}\tau^-_m + \mathrm{H.c.}) + \frac{U}{4}\sum_m \sigma^z_m\tau^z_m, \tag{2}$$

with

$$C_{m\uparrow} = (\sigma^z_1\cdots\sigma^z_{m-1})\sigma^-_m, \quad C_{m\downarrow} = \left(\prod_{m=1}^{N}\sigma^z_m\right)(\tau^z_1\cdots\tau^z_{m-1})\tau^-_m.$$

The sum over m is from 1 to N and periodic boundary conditions are assumed.

If we set $U=0$, Eq. (2) describes a pair of XY models, each of which commutes with the transfer matrix of a six-vertex model. By continuity in U, we expect that H could be related to a problem of a pair of six-vertex models coupled to each other in a suitable manner. Indeed it was shown by use of Trotter's formula that H is the logarithmic derivative of T, the transfer matrix of a certain coupled arrow vertex model.[7] At this stage there exists an ambiguity; several models may be constructed such that H is the logarithmic derivative of T (corresponding to different breakups of H). In order to resolve this ambuguity, we constructed a commuting current j by an independent argument (the details will be reported elsewhere), where

$$j = i\sum_m[\sigma^-_{m+1}\sigma^z_m\sigma^+_{m-1} - \mathrm{H.c.}] + i\frac{U}{2}\sum_m(\sigma^+_{m+1}\sigma^-_m - \sigma^+_m\sigma^-_{m+1})\times(\tau^z_m + \tau^z_{m+1}) + (\sigma \leftrightarrow \tau). \tag{3}$$

The reader may verify directly that j commutes with H. The current (with $U=0$) is in fact related simply to the second derivative of the transfer matrix for the free-Fermi six-vertex model with reference to the spectral parameter.[8] We next ask the question, can the form of the transfer matrix be deduced if we demand that its first and second derivatives with reference to a spectral parameter be reducible (apart from trivial factors) to forms involving H and j only? A detailed analysis suggests the statistical model described below.

Consider a model of two zero-field six-vertex models corresponding to arrows on the solid and dashed square lattices in Fig. 1. At each of the vertices (solid-solid and dashed-dashed) allow for six vertices obeying the ice rule (the three configurations shown in Fig. 1 and three obtained by reversal of all arrows) with Boltzmann weights a, b, and c. The solid circles are diagonal vertices, coupling the two models by giving a weight e^{h} (e^{-h}) to parallel (antiparallel) arrows. The transfer matrix connecting a row of vertical arrows to an adjacent one may be written compactly in the form

$$T=\mathrm{Tr}_0(L_{N,0}L_{N-1,0}\cdots L_{1,0}) \tag{4}$$

with

$$L_{n,0}=I_0S_{n,0}\times T_{n,0}I_0,\quad I_0=\cosh(h/2)+\sinh(h/2)\sigma_0^z\tau_0^z,$$
$$S_{n,0}=(a+b)/2+[(a-b)/2]\sigma_n^z\sigma_0^z+c(\sigma_n^+\sigma_0^-+\text{H.c.}). \tag{5}$$

$T_{n,0}$ is of the same form as $S_{n,0}$ with τ's replacing σ's. Thus at each site n we have two sets of Pauli matrices (σ's and τ's) as in the Hamiltonian, and the trace in (4) is over the row of horizontal arrows. The commutator of T with H can be written as

$$[T,H]=\sum_n \mathrm{Tr}_0(L_{N,0}\cdots[L_{n+1,0}L_{n,0},H_{n,n+1}]\cdots L_{1,0}), \tag{6}$$

with

$$H_{n,n+1}=(\sigma_n^+\sigma_{n+1}^-+\text{H.c.})+(\sigma\leftrightarrow\tau)+\frac{U}{8}(\sigma_n^z\tau_n^z+\sigma_{n+1}^z\tau_{n+1}^z).$$

Setting $R_{n,n+1,0}=[L_{n+1,0}L_{n,0},H_{n,n+1}]$, we have

$$R_{n,n+1,0}=G_{n,n+1,0}-G_{n+1,n,0}^{\dagger}, \tag{7}$$

where $G_{n,n+1,0}=L_{n,0}H_{n,n+1}L_{n,0}^{-1}$. For the "free Fermi" case the parameters a, b, and c obey the condition $c^2=a^2+b^2$, and if we restrict ourselves to this class, then the right-hand side of Eq. (7) has a separation of variables into the form $M_{n,0}^{\dagger}-M_{n+1,0}$ with

$$M_{n,0}=-\left(\frac{c}{a}\right)[(\sigma_0^+\sigma_n^-+\text{H.c.})+(\sigma\leftrightarrow\tau)]I_0^{-2}+\frac{U}{8}[L_{n,0}^{-1}\sigma_n^z\tau_n^zL_{n,0}-\sigma_n^z\tau_n^z]. \tag{8}$$

Defining $Q_{n,0}=L_{n,0}M_{n,0}$, we find

$$R_{n,n+1,0}=L_{n+1,0}Q_{n,0}^{\dagger}-Q_{n+1,0}L_{n,0}. \tag{9}$$

In the case of the XYZ model, the commutator with the transfer matrix of the eight-vertex model has the form of Eq. (9) but with $Q^{\dagger}=Q$. The commutator $[T,H]$ vanishes on summation over n. In this problem, Q is not Hermitean and hence one needs to investigate its structure further. Writing $Q_{n,0}^{\dagger}=A_{n,0}+B_{n,0}$ and $Q_{n,0}=A_{n,0}-B_{n,0}$, we find

$$R_{n,n+1,0}=(L_{n+1,0}A_{n,0}-A_{n+1,0}L_{n,0})+(L_{n+1,0}B_{n,0}-B_{n+1,0}L_{n,0}). \tag{10}$$

We may express

$$Q_{n,0}=\frac{U}{8}[\sigma_n^z\tau_n^z,L_{n,0}]+I_0D_{n,0}I_0^{-3},$$

where

$$D_{n,0}=\left(-\frac{c}{a}\right)\left(b\frac{\partial}{\partial c}+c\frac{\partial}{\partial b}\right)L_{n,0}^{(0)} \tag{11}$$

with $L_{n,0}^{(0)}=S_{n,0}\times T_{n,0}$. This yields

$$(I_0^{-1}B_{n,0}I_0^{-1})\frac{2}{\sinh(2h)}=g[L_{n,0}^{(0)},\sigma_n^z\tau_n^z]+[D_{n,0},\sigma_0^z\tau_0^z]$$

with $g=U/[4\sinh(2h)]$. A calculation shows that the right-hand side is proportional to $[L_{n,0}^{(0)}, \sigma_0^z\tau_0^z]$ for a choice of the parameter g. With

$$U = 2\sinh(2h)\frac{c^2}{ab}, \qquad (12)$$

we have

$$B_{n,0} = -\left(\frac{c^2+2b^2}{4ab}\right)(2h)[L_{n,0}, \sigma_0^z\tau_0^z]. \qquad (13)$$

Combining with Eq. (10) we find

$$R_{n,n+1,0} = (L_{n+1,0}A_{n,0} - A_{n+1,0}L_{n,0}) + [L_{n+1,0}L_{n,0}, N_0] \qquad (14)$$

with $N_0 = [(c^2+2b^2)/4ab]I_0^{-4}$. Substituting in Eq. (6) and summing over n, we find $[T,H]=0$.

In summary, we have shown that the transfer matrix T [Eq. (6)] commutes with H proved the three parameters b/a, c/a, and h obey the two constraints, $1+b^2/a^2 = c^2/a^2$ and Eq. (12). Thus we have a one-parameter family of transfer matrices commuting with H. One scheme for parametrizing the weights is to set $a=1$, $b=\sinh(t)$, $c=\cosh(t)$. An expansion of the transfer matrix in powers of t yields

$$T(t) = T(0)\left[1 + tH + \frac{t^2}{2}H^2 + \frac{t^2}{2}(-i)j + O(t^3)\right],$$

where $T(0)$ is the right-shift operator. The coefficients of powers of t in the expansion are the infinite currents commuting with H.

We have not succeeded in showing that the transfer matrices corresponding to different values of t themselves commute. This result requires an elaborate analysis, and will be presented separately.

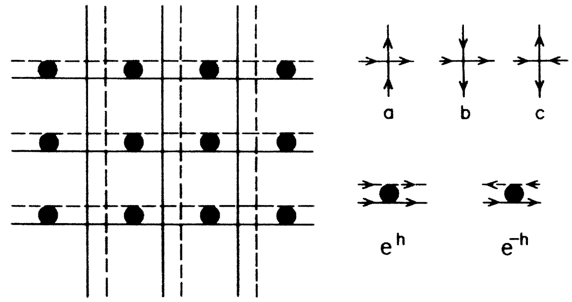

FIG. 1. The model is defined by placing arrows on the lattice shown. The distinct vertex weights are a, b, c, and e^h corresponding to configurations shown.

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