# Mathematical Models of Hysteresis

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A new approach to Preisach's hysteresis model, which emphasizes its phenomenological nature and mathematical generality, is briefly described. Then the theorem which gives the necessary and sufficient conditions for the representation of actual hysteresis nonlinearities by Preisach's model is proven. The significance of this theorem is that it establishes the limits of applicability of this model.

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Almost fifty years ago, the German physicist Preisach proposed a model of magnetic hysteresis.[1] It was based on some hypotheses concerning the physical mechanisms of magnetization. This model was primarily known in the area of magnetics where it originated much discussion (see, for instance, Néel,[2] Woodward and Della Torre,[3] Brown,[4] and Barker *et al.*[5]). In the 1970's, the Russian mathematician Krasnoselskii came across Preisach's model and understood that it contained a new general mathematical idea. Krasnoselskii separated this model from its physical meaning and represented it in a pure mathematical form which is similar to a spectral resolution of operators.[6] As a result, a new mathematical tool has evolved which can now be used for the mathematical description of hysteresis of any physical nature. At the same time, Krasnoselskii's approach has strongly revealed the phenomenological nature of Preisach's model. This, in turn, has led to the problem of determining the conditions under which actual hysteresis nonlinearities can be represented by this model. In this paper, I first briefly describe the essence of Krasnoselskii's approach, and then formulate and prove the theorem which gives necessary and sufficient conditions for the representations of actual hysteresis nonlinearities by Preisach's model. The significance of this theorem is that it establishes the limits of applicability of Preisach's model.

Consider a transducer which can be characterized by an input $u(t)$ and an output $f(t)$. This transducer is called a hysteresis transducer if its input-output relationship is a multibranch nonlinearity for which a branch-to-branch transition occurs after each input extremum. Only the case of a static hysteresis nonlinearity will be further discussed. The term "static" means that branches of hysteresis nonlinearities are determined by sequences of input extrema, while the speed of input variation between extremum points has no influence on branching.

Usually, the hysteresis transducer is a part of a system. Consequently, its input is not known beforehand but is determined by the interaction of the transducer with the rest of the system. For this reason, a mathematical model is needed which itself (because of its structure) will detect and accumulate input extrema and will choose the appropriate branch of the hysteresis nonlinearity with respect to the accumulated history. By using such models one can attempt mathematical descriptions of systems with hysteresis. Such models can be constructed by use of the Preisach-Krasnoselskii approach which is described below.

Consider an infinite set of simplest hysteresis operators $\hat{\gamma}_{\alpha\beta}$. These operators can be represented by rectangular loops on the input-output plane. Numbers $\alpha$ and $\beta$ correspond to "up" and "down" switching values of input. Outputs of these operators may assume only two values, $+1$ and $-1$. Each of these operators has a local memory. This means that given the output $f(t_0)$ at instant $t_0$ and the input $u(t)$ at all subsequent instants of time $t \geqslant t_0$, then the output $f(t)$ is uniquely determined for all $t \geqslant t_0$. In other words, in the case of local memory, the past exerts its influence upon the future through instantaneous values of output. Along with the set of operators $\hat{\gamma}_{\alpha\beta}$, consider an arbitrary weight function, $\mu(\alpha,\beta)$. Then the Preisach-Krasnoselskii model is given by

$$f(t) = \hat{\Gamma} u(t) = \iint_{\alpha \geqslant \beta} \mu(\alpha,\beta) \hat{\gamma}_{\alpha\beta} u(t)\, d\alpha\, d\beta. \qquad (1)$$

In the sequel, the case when $\mu(\alpha,\beta)$ is a finite function with a support within some triangle $T$ will be discussed. This case includes the important class of nonlinearities with limiting hysteresis loops (envelopes).

The investigation of model (1) is considerably facilitated by its geometric interpretation. This interpretation is based on the fact that there is a one-to-one correspondence between operators $\hat{\gamma}_{\alpha\beta}$ and points $(\alpha,\beta)$ of the half-plane $\alpha \geqslant \beta$. By use of this fact, it can be concluded that, at any instant of time, the triangle $T$ is subdivided into two sets (see Fig. 1): $S^+(t)$ consisting of points $(\alpha,\beta)$ for which $\hat{\gamma}_{\alpha\beta} u(t) = 1$, and $S^-(t)$ consisting of points $(\alpha,\beta)$ for which $\hat{\gamma}_{\alpha\beta} u(t) = -1$. It can be shown that the interface $L(t)$ between $S^+(t)$ and $S^-(t)$ is a staircase line whose vertices have $\alpha$ and $\beta$ coordinates coinciding

 

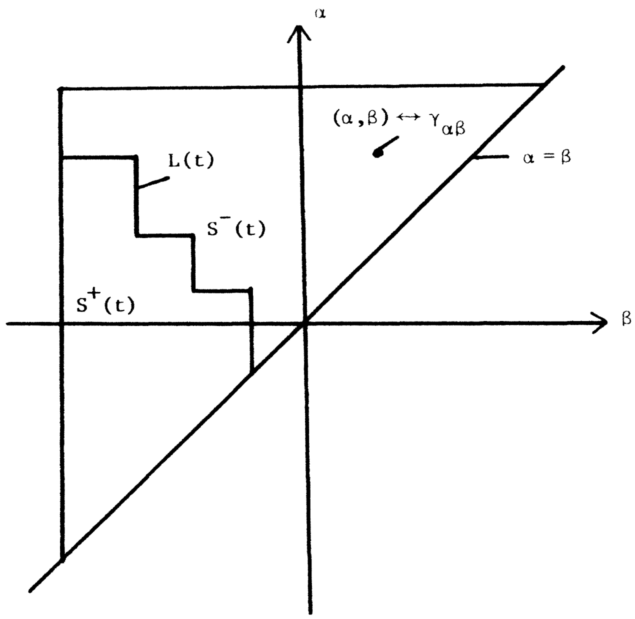


FIG. 1. A geometric interpretation of the model.

with local maxima and minima of input at previous instants of time. The final link of $L(t)$ is attached to the line $\alpha=\beta$ and moves when the input changes. This link is a horizontal one and moves up when the input increases, and it is a vertical one and moves from right to left when the input decreases. By use of the above geometric interpretation, the model (1) can be represented in the following equivalent form:

$$f(t)=\iint_{S^+(t)}\mu(\alpha,\beta)d\alpha\,d\beta-\iint_{S^-(t)}\mu(\alpha,\beta)d\alpha\,d\beta. \tag{2}$$

From expression (2), it follows that an instantaneous value of output depends on the shape of the interface $L(t)$, which in turn is determined by the extremum values of input at previous instants of time. Consequently, the past extremum values of input shape the interface $L(t)$, and in this way they leave their mark upon the future. Thus, the model (1) has a nonlocal memory. It is remarkable that a superposition of simplest hysteresis operators $\hat{\gamma}_{\alpha\beta}$ with local memories results in a hysteresis operator (1) with a new qualitative property: nonlocal memory.

It turns out that not all extremum input values are accumulated by the model (1); some of them can be wiped out. More precisely, it can be shown [by use of the rules of motion of the final link of $L(t)$] that the following fact is valid.

*Property A (wiping-out property).*—Each local maximum wipes out the vertices whose $\alpha$ coordinates are below this maximum, and each local minimum wipes out the vertices whose $\beta$ coordinates are above this minimum.

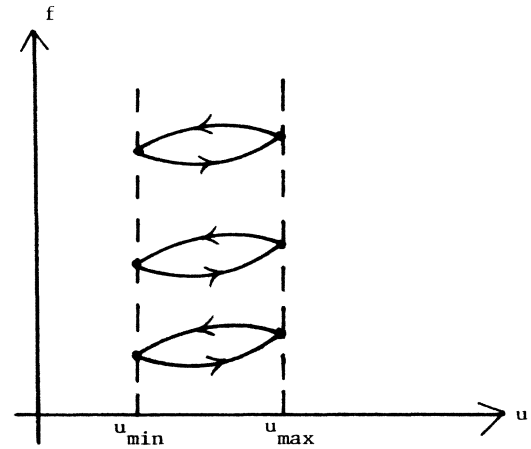


FIG. 2. Congruent hysteresis loops.

It is clear that the wiping out of vertices is equivalent to the erasing of the history associated with these vertices. Thus, subsequent variations of input might erase some previous history.

It is worth noting that the above property is natural and consistent with some experimental facts. Indeed, experiments in the area of magnetics show the existence of limiting hysteresis loops (envelopes) whose shapes do not depend on how these limiting loops are approached. It means that any past history can be wiped out by input oscillations of sufficiently large magnitude, which is in agreement with property A. The wiping-out property can also be linked to the recently discovered magnetic Kaiser effect.[7]

Another characteristic property of the model (1) is illustrated by Fig. 2. It can be stated as follows.

*Property B (congruency property).*—All hysteresis loops corresponding to the same extremum values of input are congruent in the geometrical sense. The proof of this property can be easily obtained from the fact that for input variations within the same range, the final links of the staircase interfaces will move identically within the same triangles. This will result in equal output increments, which is tantamount to the congruency of hysteresis loops.

Now, we can formulate the fundamental result.

*Theorem.*—Properties A and B constitute necessary and sufficient conditions for a hysteresis transducer to be represented by the model (1) on the set of piecewise monotonic inputs.

*Proof.*—Necessity: Let a hysteresis transducer be represented by the model (1). Then, this transducer should have the same properties as this model; in particular, it should have the properties A and B.

Sufficiency: The proof of this part is constructive. First, the weight function, $\mu(\alpha,\beta)$, will be found for the given transducer with use of some experimental data obtained for this transducer. Next, it will be proven that for the above weight function, the model (1) and the given transducer have the same input-output relationships provided that properties A and B hold for the transducer.

To determine $\mu(\alpha,\beta)$, the set of first-order transition (reversal) curves should be experimentally found. This can be done by our first bringing the input to such a value that outputs of all operators $\hat{\gamma}_{\alpha\beta}$ are equal to $-1$. If we now gradually increase the input value, then we will follow along a limiting ascending branch (see Fig. 3). This branch is called limiting because there is no branch below it. The notation $f_\alpha$ will be used for the output value on this branch corresponding to the input $u=\alpha$. The first-order transition curves are attached to the limiting ascending branch. Each of these curves is formed when the above monotonic increase of the input is followed by a subsequent monotonic decrease. For this reason, these curves can be called first-order decreasing transition curves. The notation $f_{\alpha\beta}$ will be used for the output value on the transition curve attached to the limiting ascending branch at the point $f_\alpha$. This output value corresponds to the input $u=\beta$. Now we can define the function

$$F(\alpha,\beta)=\tfrac{1}{2}(f_\alpha-f_{\alpha\beta}). \tag{3}$$

By use of the geometric interpretation of the model (1), it is easy to prove that

$$F(\alpha,\beta)=\iint_{T(\alpha,\beta)}\mu(x,y)\,dx\,dy =\int_\beta^\alpha\left[\int_\beta^y \mu(x,y)\,dx\right]dy, \tag{4}$$

where $T(\alpha,\beta)$ is the triangle formed by the intersection of the lines $x=\beta$, $y=\alpha$, and $y=x$.

From (4), we find that

$$\mu(\alpha,\beta)=-\partial^2F(\alpha,\beta)/\partial\beta\,\partial\alpha. \tag{5}$$

We have used the first-order decreasing transition curves to determine $\mu(\alpha,\beta)$. But for the same purpose and in a similar manner, we could use the first-order increasing transition curves which are attached to the limiting descending branch. These curves can be parametrized as $f_{\beta'\alpha'}$ (see Fig. 3). On physical grounds (symmetry considerations), it is clear that curves $f_{\alpha\beta}$ are congruent to curves $f_{\beta'\alpha'}$ if $\beta'=-\alpha$. By use of this fact and (3)–(5), it can be easily proved that

$$\mu(\alpha,\beta)=\mu(-\beta,-\alpha). \tag{6}$$

Thus, either the first-order increasing or the first-order decreasing transition curves can be used for the determination of $\mu(\alpha,\beta)$.

Now, I will prove that if $\mu(\alpha,\beta)$ is substituted from (5) into (1), then the model (1) and the transducer will have the same input-output relationships. This is true for the first-order transition curves because of the very way in which $\mu(\alpha,\beta)$ was determined. Next, the induction argument will be used. Let us assume that the above statement is true for a transition curve number $k$, and then I will prove that this statement holds for a transition curve number $k+1$.

Let $a$ be a point at which the transition curve number $k+1$ starts (see Fig. 4). This point corresponds to some input value $u=\alpha$. According to the induction assumption, the output values of the transducer and the model (1) coincide at the point $a$. Consequently, it remains to be proven that the output in-

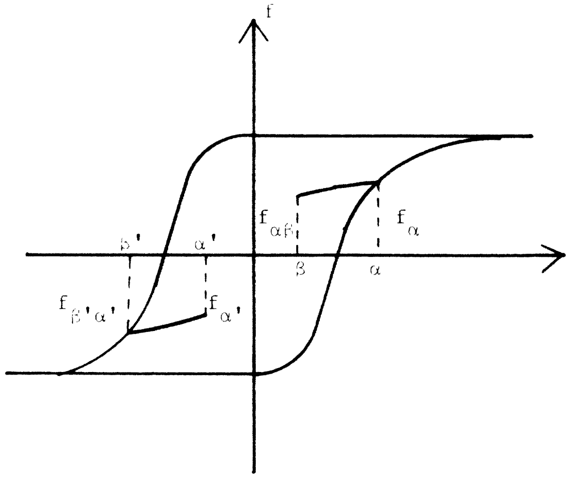


FIG. 3. Congruency of first-order transition curves $f_{\alpha\beta}$ and $f_{\beta'\alpha'}$.

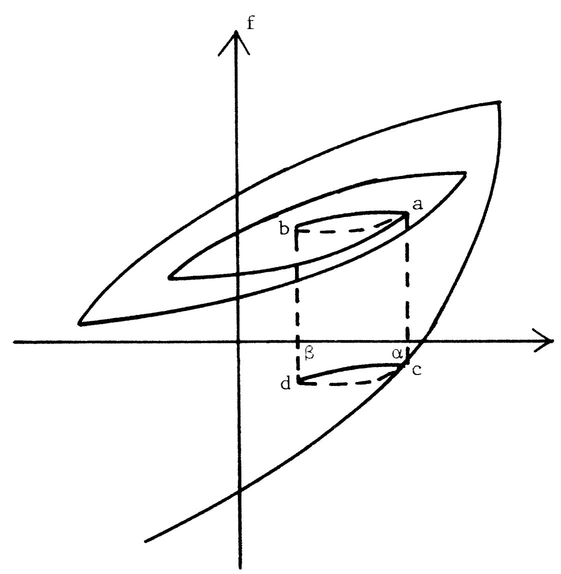


FIG. 4. Geometric illustration of the proof of sufficiency.

crements along the transition curve number $k+1$ are the same for the actual transducer and for the model (1). Consider an arbitrary input value $u=\beta<\alpha$. The output increment for the transducer will be equal to the increment of $f$ along the curve $ab$ (Fig. 4). By use of the geometric interpretation of the model (1), it is easy to show that the corresponding output increment for the model is given by

$$\Delta f = -2 \iint_{T(\alpha,\beta)} \mu(x,y)\,dx\,dy. \tag{7}$$

But, according to (4) and (3), this increment is equal to the increment of $f$ along the first-order transition curve $cd$ (Fig. 4). Thus, it remains to be shown that the output increments along the curves $ab$ and $cd$ are the same. It is here that properties A and B will be used. The proof proceeds as follows. If starting from the point $b$ we monotonically increase the input value from $\beta$ back to $\alpha$, then, according to property A, we will arrive at the point $a$. In other words, we will move along some curve $ba$ which is below the curve $ab$. On the other hand, if starting from the point $d$ we monotonically increase the input value from $\beta$ to $\alpha$, then, again, according to property A, we will arrive at the point $c$ moving along some curve $dc$. According to property B, hysteresis loops $bab$ and $dcd$ are congruent. Conseqently, the increments of $f$ along the curves $ab$ and $cd$ are the same and the proof is complete.

It is easy to see that the essence of the proof is in the reduction of higher-order transition curves to first-order transition curves. This reduction rests on properties A and B.

The same fact admits another interpretation. The experimental data provided by the first-order transition curves allow one to determine the weight function $\mu(\alpha,\beta)$. Then, if we assume properties A and B and use the model (1), higher-order transition curves can be determined. In this sense, the mathematical model (1) has prediction power.

A hysteresis phenomenon is associated with a memory. For this reason, the model (1) might have appeal as the mathematical model of memory with some interesting properties. I will discuss only a few of them. First, the mechanism of memory formation in (1) is surprisingly simple and results from the superposition (parallel connection) of qualitatively similar elements (two-position cells) $\hat{\gamma}_{ab}$. Secondly, the model (1) stores information (extremum values of input) not in particular separate cells (as in the case of computer storage devices), but some ensembles of the cells $\hat{\gamma}_{\alpha\beta}$ participate in storage of each bit of information. As a result, if some of the cells $\hat{\gamma}_{\alpha\beta}$ are destroyed, the stored information might still be preserved. The above properties are somewhat similar to those being observed (or suspected) for memories in biological systems. However, it will be imprudent to speculate now how far this similarity goes. Nevertheless, it may be expected that the mathematical tool (1) might find some applications beyond the area of hysteresis modeling.

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