PHYSICAL REVIEW LETTERS

VOLUME 56 — 14 APRIL 1986 — NUMBER 15

Input States for Enhancement of Fermion Interferometer Sensitivity

B. Yurke

AT&T Bell Laboratories, Murray Hill, New Jersey 07974

(Received 14 August 1985)

A conventional fermion interferometer, in which the fermions enter only one of the two input ports, can achieve a phase sensitivity $\Delta\phi = 1/\sqrt{n}$, where n is the total number of fermions which have passed through the interferometer. Here it is shown that by injection of fermions into both input ports the phase sensitivity can approach $1/n$ provided that the fermions in the two input beams are suitably correlated.

PACS numbers: 03.65.Bz

A conventional optical interferometer in which light enters one input port can achieve a phase sensitivity $\Delta\phi = 1/\sqrt{n}$, where n is the total number of photons which have passed through the interferometer. It has been shown by Caves[1] and Bondurant and Shapiro[2,3] that when squeezed states of the electromagnetic field are fed into both ports of an interferometer the interferometer's sensitivity $\Delta\phi$ can approach $1/n$. Squeezed states are many-body states in which a large number of bosons occupy a given boson mode.[4] Since at most one fermion can occupy a given fermion mode, it is not immediately apparent that fermion analogs of squeezed states can be constructed or whether fermion interferometers can also achieve a phase sensitivity approaching $1/n$. It is shown here that under quasimonochromatic conditions where a large number of states are made available to the fermions for occupancy, many-fermion states can in fact be constructed which allow fermion interferometers such as electron[5] or neutron[6] interferometers to achieve a phase sensitivity approaching $1/n$.

The interferometer considered here is depicted in Fig. 1. For simplicity it will be assumed that the fermion beams are sufficiently well collimated that they can be adequately described by a one-dimensional Schrödinger equation. The field operators for the input fields $\psi_1^{\rm in}(z,t)$ and $\psi_2^{\rm in}(z,t)$, where z is the distance along the respective beam paths, have the form

$$\psi_1^{\rm in}(z,t) = (2\pi)^{-1/2}\sum_s\int_B dk\, c_1^{\rm in}(k,s)\, e^{-i(\omega_k t - kz)}, \tag{1}$$
$$\psi_2^{\rm in}(z,t) = (2\pi)^{-1/2}\sum_s\int_B dk\, c_2^{\rm in}(k,s)\, e^{-i(\omega_k t - kz)},$$

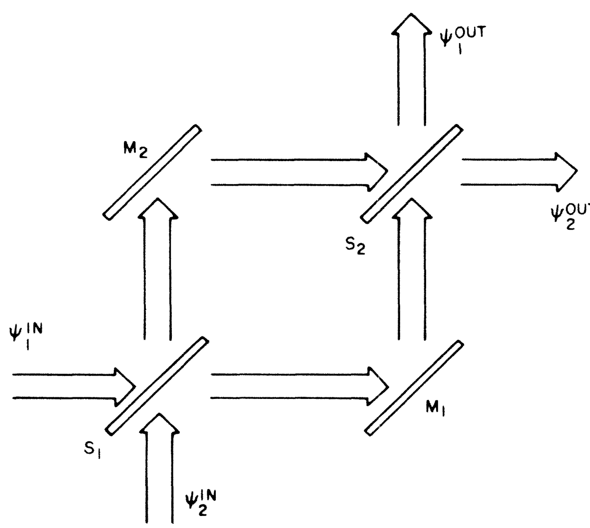

FIG. 1. An interferometer in the Mach-Zehnder configuration. S_1 and S_2 are beam splitters; M_1 and M_2 are mirrors. Note that the interferometer has two input ports and two output ports.

where

$$\omega_k = \hbar k^2/2m \tag{2}$$

and the fermion annihilation operators $c_a^{\text{in}}(k,s)$ satisfy the usual anticommutation relations, in particular,

$$[c_a^{\text{in}}(k,s), c_b^{\text{in}\dagger}(k',s')]_+ = \delta_{ab}\delta_{ss'}(k-k'). \tag{3}$$

The fermions are assumed to have a finite energy spread or, equivalently, a finite spread ΔK in k. The range B of the k integration may thus be restricted to $k_0 - \Delta k/2 < k < k_0 + \Delta k/2$. The field operators for the output beams are similarly given by

$$\psi_1^{\text{out}}(x,t) = (2\pi)^{-1/2}\sum_s \int_B dk\, c_1^{\text{out}}(k,s)e^{-i(\omega_k t - kx)},\quad \psi_2^{\text{out}}(x,t) = (2\pi)^{-1/2}\sum_s \int_B dk\, c_2^{\text{out}}(k,s)e^{-i(\omega_k t - kx)}, \tag{4}$$

If we choose the coordinate z to be 0 at mirror M_1 and x to be 0 at mirror M_2, the mode transformation performed by the interferometer may be taken to be[6]

$$\begin{pmatrix} c_1^{\text{out}}(k,s) \\ c_2^{\text{out}}(k,s) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{ikl_1} + e^{ikl_2} & i(e^{ikl_1} - e^{ikl_2}) \\ -i(e^{ikl_1} - e^{ikl_2}) & e^{ikl_1} + e^{ikl_2} \end{pmatrix}\begin{pmatrix} c_1^{\text{in}}(k,s) \\ c_2^{\text{in}}(k,s) \end{pmatrix}, \tag{5}$$

where l_1 and l_2 denote the path lengths of the beams which propagate between S_1 and S_2 via mirrors M_1 and M_2, respectively. With Eq. (5), operators representing physical quantities measured at the output of the interferometer can be expressed in terms of creation and annihilation operators of the input modes. This facilitates calculation since the moments of the operators can then be evaluated by taking expectation values with respect to the input states. In particular, the number operator for the total number of particles leaving the output ports of the interferometer, expressed in terms of the input mode creation and annihilation operators, is

$$N = \sum_{s=1}^{2}\int_B dk\,[c_1^{\text{in}\dagger}(k,s)c_1^{\text{in}}(k,s) + c_2^{\text{in}\dagger}(k,s)c_2^{\text{in}}(k,s)]. \tag{6}$$

Similarly, the operator N_D for the difference in the numbers of particles leaving the two output ports, expressed in terms of the input mode creation and annihilation operators, is

$$N_D = \sum_s \int_B dk \cos[k(l_1 - l_2)][c_1^{\text{in}\dagger}(k,s)c_1^{\text{in}}(k,s) - c_2^{\text{in}\dagger}(k,s)c_2^{\text{in}}(k,s)] \\ - \sum_s \int_B dk \sin[k(l_1 - l_2)][c_1^{\text{in}\dagger}(k,s)c_2^{\text{in}}(k,s) - c_2^{\text{in}\dagger}(k,s)c_1^{\text{in}}(k,s)]. \tag{7}$$

At this point it is convenient to introduce the abstract operators

$$\begin{aligned} J_x &= \tfrac{1}{2}\sum_s \int_B dk\,[c_1^{\text{in}\dagger}(k,s)c_2^{\text{in}}(k,s) + c_2^{\text{in}\dagger}(k,s)c_1^{\text{in}}(k,s)], \\ J_y &= -\tfrac{1}{2}i\sum_s \int_B dk\,[c_1^{\text{in}\dagger}(k,s)c_2^{\text{in}}(k,s) - c_2^{\text{in}\dagger}(k,s)c_1^{\text{in}}(k,s)], \\ J_z &= \tfrac{1}{2}\sum_s \int_B dk\,[c_1^{\text{in}\dagger}(k,s)c_1^{\text{in}}(k,s) - c_2^{\text{in}\dagger}(k,s)c_2^{\text{in}}(k,s)], \end{aligned} \tag{8}$$

which satisfy the usual angular momentum commutation relations.[7] As will be shown, these operators allow one to take advantage of angular momentum algebra or SU(2) representation theory. The $\mathbf{J} = (J_x, J_y, J_z)$ commutes with the number operator N:

$$[N, \mathbf{J}] = 0. \tag{9}$$

Further, the Casimir invariant $J^2 = J_x^2 + J_y^2 + J_z^2$ can be put into the form

$$J^2 = \tfrac{1}{2}N(\tfrac{1}{2}N + 1) + W, \tag{10}$$

where

$$W = -\sum_s \sum_{s'} \int_B dk \int_B dk'\,[c_1^{\text{in}\dagger}(k,s)c_2^{\text{in}\dagger}(k',s')c_2^{\text{in}}(k,s)c_1^{\text{in}}(k',s') + c_1^{\text{in}\dagger}(k,s)c_2^{\text{in}\dagger}(k',s')c_2^{\text{in}}(k',s')c_1^{\text{in}}(k,s)] \tag{11}$$

commutes with $\mathbf{J}$:

$$[\mathbf{J}, W] = 0. \tag{12}$$

When the beam is quasimonochromatic, $\Delta k(l_1 - l_2) \ll 1$, $\cos[k(l_1 - l_2)]$ and $\sin[k(l_1 - l_2)]$ may be approximated as $\cos[k_0(l_1 - l_2)]$ and $\sin[k_0(l_1 - l_2)]$. With this approximation N_D takes the form

$$N_D = 2\cos\phi J_z - 2\sin\phi J_x, \tag{13}$$

where $\phi = k_0(l_1 - l_2)$. With the operator for the difference in the number of particles leaving the two output ports expressed in terms of J_x and J_z, SU(2) representation theory will now be exploited to determine interferometer sensitivity for various input states.

It will first be demonstrated that when fermions enter the inteferometer through only one port the phase sensitivity $\Delta\phi$ is at best $1/\sqrt{n}$. Suppose that ideal particle counters are located at the output ports and that the fermions enter port 1 of the interferometer. Since no fermions enter port 2 it is evident from Eq. (11) that the input state is in an eigenstate of W with an eigenvalue of zero. After the fermions have passed through the interferometer the detectors report that a total of n particles have passed through the interferometer. From Eq. (10) one deduces that the input state is an eigenstate of J with eigenvalue $j=n/2$. Since all n photons enter through the input port 1, it is evident from Eq. (8) that the input state is an eigenstate of J_z with eigenvalue $m=n/2$. Hence the input state is of the form $|j,m\rangle$, where $j=m=n/2$. The expectation values of the moments of N_D with respect to this state can readily be evaluated by standard techniques of angular momentum algebra; in particular,

$$\langle N_D\rangle = n\cos\phi, \quad (\Delta N_D)^2 = n\sin^2\phi. \tag{14}$$

The rms fluctuation $\Delta\phi$ in the inferred phase ϕ for an ensemble of such measurements is determined by

$$(\Delta\phi)^2 = \frac{(\Delta N_D)^2}{|\partial\langle N_D\rangle/\partial\phi|^2}. \tag{15}$$

Substituting (14) into (15) one readily obtains a phase sensitivity

$$\Delta\phi = 1/\sqrt{n}. \tag{16}$$

It is worth emphasizing the generality of this result. Nothing was assumed about how the n fermions were distributed in the k-space interval $[k_0 - \frac{1}{2}\Delta k, k_0 + \frac{1}{2}\Delta k]$ or in the spin space.

An example of a state exhibiting a phase sensitivity approaching $1/n$ will now be constructed. By application of the lowering operator $J_- = J_x - iJ_y$,

$$J_- = \sum_s \int_B dk\, c_2^{\mathrm{in}\dagger}(k,s) c_1^{\mathrm{in}}(k,s), \tag{17}$$

repeatedly to the state $|j,m\rangle = |\frac{1}{2}n,\frac{1}{2}n\rangle$ the states $|\frac{1}{2}n,\frac{1}{2}\rangle$ and $|\frac{1}{2}n,-\frac{1}{2}\rangle$ can be constructed provided n is odd. With each application of the lowering operator a fermion is transferred from input beam 1 to input beam 2. Hence for the state $|\frac{1}{2}n,\frac{1}{2}\rangle$ there are $(n+1)/2$ fermions in beam 1 and $(n-1)/2$ fermions in beam 2. Consider now the state

$$|S\rangle = 2^{-1/2}[\,|\tfrac{1}{2}n,\tfrac{1}{2}\rangle + |\tfrac{1}{2}n,-\tfrac{1}{2}\rangle\,]. \tag{18}$$

The expectation value of N_D and its variance are for this state

$$\begin{aligned}\langle N_D\rangle &= \tfrac{1}{2}[n+1]\sin\phi,\\ (\Delta N_D)^2 &= \cos^2\phi + \tfrac{1}{4}(n^2+n-1)\sin^2\phi.\end{aligned} \tag{19}$$

The mean square uncertainty in ϕ is then from Eq. (15)

$$(\Delta\phi)^2 = \frac{\cos^2\phi + \frac{1}{4}(n^2+n-1)\sin^2\phi}{\frac{1}{4}(n+1)^2\cos^2\phi}. \tag{20}$$

This is minimized when $\sin\phi = 0$. Hence the minimum rms uncertainty in ϕ for this state is

$$\Delta\phi_{\min} = 2/(n+1). \tag{21}$$

It has now been shown that fermion states can be constructed which allow the interferometer to achieve a phase sensitivity which for large n has the form α/n, where α is a constant of order unity. This phase sensitivity is achieved only for phase angles near those satisfying $\sin\phi = 0$. The particular form, Eq. (18), is not essential for achieving a phase sensitivity approaching α/n and n need not be odd. Generally speaking, if one forms a superposition of neighboring states $|\frac{1}{2}n,m\rangle$ lying near $m=0$, such states will exhibit a phase sensitivity approaching $1/n$, provided some care is exercised in the phasing of the states with respect to each other in the superposition. With such states, for which $\langle J_z\rangle$ is small, roughly equal numbers of fermions enter each port of the interferometer but in a highly correlated manner.

Here I do not offer any means by which such states may be generated in the laboratory. It is worth pointing out, however, that considerable experimental progress has been made in generating boson states, squeezed states, with analogous properties[7] by use of four-wave mixers.[8-10]

I would like to thank S. L. McCall, J. R. Klauder, R. E. Slusher, and J. S. Denker for stimulating discussions on this topic.

[1]C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

[2]R. S. Bondurant and J. H. Shapiro, Phys. Rev. D **30**, 2548 (1984).

[3]R. S. Bondurant, Ph.D. thesis, Massachusetts Institute of Technology, 1983 (unpublished).

[4]H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[5]S. Olariu and I. Iovitzu Popescu, Rev. Mod. Phys. **57**, 339 (1985).

[6]D. M. Greenberger and A. W. Overhauser, Rev. Mod. Phys. **51**, 43 (1979).

[7]A detailed discussion of the use of the rotation group SU(2) and the Lie group SU(1,1) in the analysis of interferometers, by B. Yurke, S. L. McCall, and J. R. Klauder, will be published elsewhere.

[8]R. E. Slusher, L. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985).

[9]M. W. Maeda, P. Kumar, and J. H. Shapiro, Topical Meeting on Instabilities and Dynamics of Lasers and Nonlinear Optical Systems, 18–21 June 1985, postdeadline paper WD30 (unpublished).

[10]M. D. Levenson, R. M. Shelby, and S. M. Perlmutter, Opt. Lett. **10**, 514 (1985).