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Novel Relativistic Effect Important in Accelerators

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It is shown that a bunch of charged particles following a curved path in a magnetic field is subject to a force due to its own electromagnetic field. One aspect of this is a "centrifugal" force acting on individual particles in the bunch. A resonance mechanism, capable of disrupting the beam at modest currents, is given as an example of the importance of this force. The theory is tested with observations from the Cornell Electron Storage Ring. This force will cause important modifications to existing theories of accelerator stability.

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For charged-particle beams in circular accelerators at high energies it has been customary to neglect space-charge forces because of the near perfect cancellation between electric and magnetic forces. By this assumption any current-dependent force can only result from cavities, discontinuities, resistive walls,[1] or other nearby material. Here it will be shown that there is an important deflecting force, related to synchrotron radiation, which is present even in the absence of such material. Nearby walls do, however, influence the force appreciably.

Assume initially that moving charges are uniformly distributed with line density λ along a circle C of radius R. We will calculate the force on any one of these charges and then argue that almost the same formula can be applied to not-too-short bunched beams by allowing the local line density $\lambda(s_B)$ to depend on s_B, the distance from the center of the bunch. Like all quantities in this paper, s_B is measured in the laboratory frame.

The fields acting on a unit charge at a field point P due to the presence of charge $\lambda\, ds_B$ at a source point P' at an earlier time are given by the Lienard-Wiechert formulas[2]

$$d\mathbf{E}=\frac{\lambda\, ds_B}{4\pi\epsilon_0}\left(\frac{\hat{\mathbf{n}}-\mathbf{v}'}{\gamma^2 D^3 l^2}+\frac{\hat{\mathbf{n}}\times\{(\hat{\mathbf{n}}-\mathbf{v}')\times\mathbf{a}'\}}{D^3 l}\right), \qquad (1)$$

$$d\mathbf{B}=\hat{\mathbf{n}}\times d\mathbf{E}, \qquad (2)$$

where

$$D=1-\hat{\mathbf{n}}\cdot\mathbf{v}'=1-\beta\cos\tfrac{1}{2}\theta \cong \gamma^{-2}/2+(\theta/2)^2/2. \qquad (3)$$

With s being the arc length from source point to field point, the quantities $\theta=s/R$, l, and $\hat{\mathbf{n}}$ are respectively the angle, distance, and unit vector from P' to P. With $c=1$ the velocities at P' and P and $\mathbf{v}'$ and $\mathbf{v}$ and β and γ have their usual relativistic meanings. Typical numerical values are given in Table I.

The second term of (1), being proportional to the acceleration $a'=v^2/R$, is the essential new ingredient present because the path is curved. Since $\mathbf{v},\mathbf{v}'$, and $\hat{\mathbf{n}}$ are all more or less parallel and because the field at P is approximately a transverse plane wave the electric and magnetic forces approximately cancel. In the interest of handling this cancellation explicitly, we introduce small quantities δ, δ', $\boldsymbol{\Delta}$, and $\boldsymbol{\Delta}'$ such that

$$\mathbf{v}=(1+\delta)\hat{\mathbf{n}}+\boldsymbol{\Delta}, \qquad (4)$$

$$\mathbf{v}'=(1+\delta')\hat{\mathbf{n}}+\boldsymbol{\Delta}', \qquad (5)$$

where

$$\boldsymbol{\Delta}\cdot\hat{\mathbf{n}}=\boldsymbol{\Delta}'\cdot\hat{\mathbf{n}}=0. \qquad (6)$$

Note that δ' is equal to $-D$.

We concentrate on the radial component dF of the force per unit charge due to these fields. It is given by

$$dF = \frac{\lambda\, ds_B}{4\pi\epsilon_0 D^3 l}\left\{\frac{1}{\gamma^2 l}[-\hat{\mathbf{n}}(\delta'+\mathbf{\Delta}\cdot\mathbf{\Delta}')+\mathbf{\Delta}'\delta]+\hat{\mathbf{n}}(\delta\delta'\hat{\mathbf{n}}\cdot\mathbf{a}'-\mathbf{\Delta}\cdot\mathbf{\Delta}'\hat{\mathbf{n}}\cdot\mathbf{a}'+\delta'\mathbf{\Delta}\cdot\mathbf{a}')+\mathbf{\Delta}'\delta\hat{\mathbf{n}}\cdot\mathbf{a}'-\mathbf{a}'\delta\delta'\right\}. \quad (7)$$

For quantitative comparison with observations this expression can be integrated numerically over actual transverse and longitudinal distributions. For purposes of discussion this can be done analytically for a beam of zero transverse extent. In the change of integration variable from s_B to θ the Jacobian factor $|ds_B/d\theta| = RD$ reduces the highly peaked structure of the integrand by one power of D, yielding

$$F = \frac{\lambda R}{4\pi\epsilon_0}\int_{\theta_{\min}}^{\theta_{\max}}\frac{\{\ \}_r\, d\theta}{D^2 l} = \frac{\lambda}{4\pi\epsilon_0 R}\left[\left(1-\frac{1}{2\gamma^2}\right)\ln\frac{\tan(\theta_{\max}/4)}{\tan(\theta_{\min}/4)}-1\right]. \quad (8)$$

The first (large) term here comes mainly from the second term of (1). The limits refer to front and back of the bunch. They are not critical in the evaluation of the second (small) term.

This expression exhibits a logarithmic divergence at $\theta = 0$, but that should not be surprising as the same divergence would appear in elementary electricity and magnetism in a calculation of the "hoop stress" of an infinitely fine charged hoop or a current loop. This divergence is due to "nearby" charges and is removed when the beam is given transverse size. This can be done analytically[3] or numerically. We do the latter and obtain results such as those shown in Fig. 1. Since the divergence is only logarithmic the results are not too sensitive to the assumptions. The value of the square bracket in (8) is about 6. But this divergence accounts for the feature of Fig. 1 that F falls off quickly as the field point P moves transversely off the orbit circle, i.e., the radial position r or the vertical position z deviates from zero.

Coherent synchrotron radiation has been considered by Schwinger[4] and screening effects of nearby walls by him and by Nodvick and Saxon.[5] They calculated only the longitudinal force while the present paper deals with the transverse force, which prevents direct comparison of the results. In its simplest form the present calculation applies to free-space orbits, but to relate to experiment it is necessary to understand the screening due to inevitable nearby walls. For this paper we limit the discussion to features for which this screening can be neglected.

From the previous discussion it can be seen that an appreciable portion of the transverse force being studied comes from nearby charges and could almost be said to be nonrelativistic in origin. The logarithmic singularity entails a rapid spatial variation over distances comparable with the transverse beam size. A consequence is that, for slender beams, the force cannot be fully screened by the presence of walls which, though nearby, are distant relative to the beam size.

For this discussion and for analyzing resonances below, it is useful to introduce the following rough parametrization of the data of Fig. 1:

$$F = (R_0 + R_1 r + R_2 r^2)(1 + Z_2 z^2). \quad (9)$$

TABLE I. Numerical values.

Symbol	Meaning or equation	Value	Unit
γ		10^4	
R	Radius of orbit	80	m
ω_0	Revolution frequency	$2\pi\times 0.39\times 10^6$	sec^{-1}
σ_L	Bunch length (s.d.)	2	cm
h	Bunch full height (uniform)	2.7	mm
q	Charge in bunch	$\frac{0.01\ \mathrm{A}}{0.39\times 10^6\ \mathrm{Hz}}$	C
α	Damping (11)	40	sec^{-1}
R_0	(9)	455	V/m
R_1	(9)	−0.6	V/m·mm
R_2	(9)	−30	$\mathrm{V/m\cdot mm^2}$
Z_2	(9)	−0.11	mm^{-2}
A_r	(10)	5	mm
A_z	(10)	1	mm
A_s	(10)	2	cm

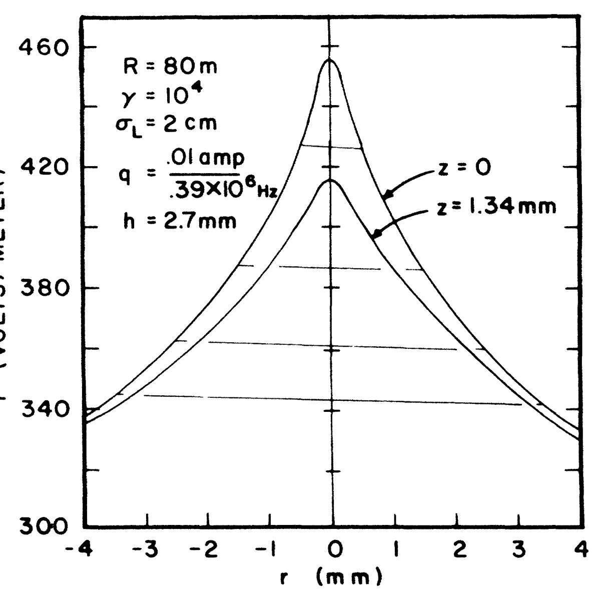

FIG. 1. Dependence of the transverse force F on radial position r and vertical position z.

This gives the transverse dependence of the force per unit charge at the bunch center longitudinally. Values of the coefficients are given in Table I. It is the coefficients R_2 and Z_2 which govern the rapid variation near the peak (due to the logarithmic singularity), which are insensitive to screening, and which will dominate the subsequent discussion. The other coefficients, strongly influenced by screening, will be unimportant.

We turn next to the observable effects of the force per unit charge F. Close to their previous meanings, but referring now to the nominal rather than the actual bunch center, the variables r, z, and $s = R\omega_0 t$ locate a particle in space and s_B locates it longitudinally within a bunch. Focusing elements cause these variables to oscillate (in the absence of perturbation) according to

$$r,z,s_B = A_{r,z,s}\cos(Q_{r,z,s}\omega_0 t), \tag{10}$$

where the frequencies are greater than the revolution frequency ω_0 by the "tune" factors Q_r, Q_z, and Q_s. With inclusion of damping α and the new force the radial equation of motion is

$$\frac{d^2r}{dt^2} + 2\alpha\frac{dr}{dt} + Q_r^2\omega_0^2 r = \frac{c^2}{pc/e}F(r,z)\exp\left(-\frac{s_B^2}{2\sigma_L^2}\right)D(s), \tag{11}$$

where e is the charge and p the momentum of the particle. The factor $D(s)$ alternates between 0 and 1 as the particle leaves and enters bending magnets. The Gaussian factor describes the longitudinal bunch profile. It oscillates as s_B oscillates. This oscillation is accompanied by an oscillation (in quadrature) of the particle's deviation in momentum from a central value.

There are "chromatic" effects accompanying such changes. A particle temporarily shifted up in momentum by Δp will be stiffer and hence more weakly focused. If we call the resultant shift in radial frequency $\omega_0\Delta Q_r$ the ratio $\Delta Q_r/\Delta p$ is called the chromaticity. At the same time such a momentum shift will cause the particle to move to larger radius, i.e., the outside of the bunch. Here the space-charge force provides a focusing effect which also shifts the frequency. A measure of the importance of this previously neglected force is that these two shifts are roughly equal for the parameters of Table I.

Since the force per unit charge F is small it can have an important effect only if, as a result of resonance, its effect is reinforced turn after turn. To analyze this one must make a Fourier decomposition of the right-hand side of (11) into terms of the form

$$\frac{c^2}{pc/e}R(n_r,n_z,n_s,n)\cos[(n_rQ_r+n_zQ_z+n_sQ_s+n)\omega_0 t], \tag{12}$$

where n_r, n_z, n_s, and n are integers of either sign connected respectively with r, z, s_B, and s. All possible sum and difference frequencies enter unless excluded by symmetry. There is driven response at each such frequency and the actual variation of r [over and above that given by (10)] is the linear superposition of all these responses. Normally at most one of these terms is important since its frequency is close to $Q_r\omega_0$, the natural frequency of the unperturbed oscillator.

Exactly on resonance the maximum response, obtained by solving (11), is given by

$$r_{\max} = \frac{c^2}{pc/e}R(n_r,n_z,n_s,n)\frac{1}{2\omega_0 Q_r\alpha}. \tag{13}$$

If this is too large the particle will be lost.

By varying Q_r, Q_z, and Q_s we have, at the Cornell Electron Storage Ring (CESR), detected various resonances of this sort with order of magnitude consistent with (13). The centrifugal space-charge force is not the only force conceivably causing this behavior,[6,7] but one whole class of competitors was ruled out experimentally by showing that these resonances disappeared for low beam currents. That is expected from the proportionality to charge density in (8).

More detailed and quantitative data will be presented for one particular resonance. After reduction of the beam current sufficiently nondestructive observation of the bunch "dynamics" was possible. It consisted of administering a small horizontal deflection and measuring the rate $1/\tau$ at which the subsequent oscillation damped. That rate is plotted in Fig. 2(a) as a function of the radial tune Q_r. The narrow resonance occurs for tune values which satisfy

$$Q_r = Q_z + Q_s. \tag{14}$$

By making small variations in any one of these tunes and compensating with the others it was unambiguously confirmed that (14) truly characterized the resonance.

To check this picture a computer simulation was written in which 500 particles were subjected to these forces for 400 turns. The result is plotted in Fig. 2(b). A similar resonance is observed. For various reasons a more quantitative comparison than is suggested by the two figures is impractical. For example, the resonance in Fig. 2(b) is as narrow as could be obtained by tracking only 500 particles, a limit dictated by computer time.

One more feature had to be incorporated into the simulation to account for this particular resonance. That feature is "coupling" between horizontal and vertical oscillations caused by skew quadrupoles in the accelerator. Analytically such coupling is described by addition of another term $k_C z$ to Eq. (11). This necessarily entails also addition of a term $k_C r$ to the corresponding equation of vertical motion. Both in the

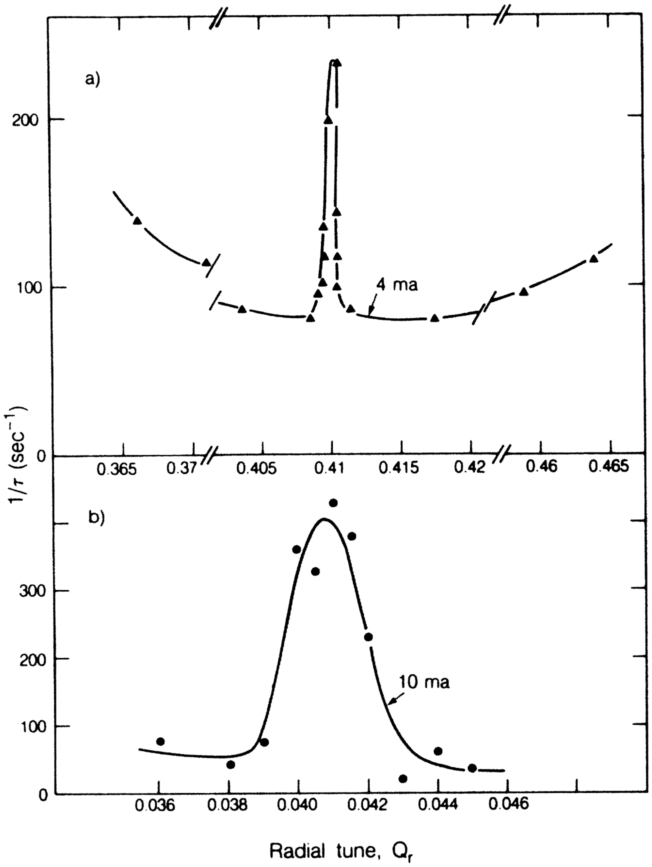

FIG. 2. (a) Damping rate $1/\tau$ measured in CESR and exhibiting a synchrobetatron resonance. (b) The same resonance investigated by a multiparticle computer simulation. Existence of the resonance depends on both the centrifugal space-charge force and coupling of vertical and horizontal motion.

CESR and in the simulation the coupling coefficient k_C is under the control of the operator and similar behavior was observed. When k_C is set equal to zero this resonance disappears. When it is turned up the data of Fig. 2 result. The coupling parameter and all other parameters in the simulation were fixed consistent with our best understanding of the accelerator operation.

It is the strong collimation of the field pattern which makes it a potent driver of such a complicated resonance as $Q_r = Q_z + Q_s$ in which all the tunes and amplitudes enter. The force varies appreciably both horizontally and vertically for distances comparable to the transverse dimensions of the beam. The fields of a cavity, having much larger transverse dimensions, tend to vary appreciably only on a larger distance scale. For this reason there are good prospects that the resonance exhibited in Fig. 2 should be completely describable by this previously neglected force and not be polluted by other beam-wall interactions. That is why it has been emphasized in this paper.

This previously neglected force is essentially different from the well-known (but hard to calculate reliably) self-forces due to transverse coupling "impedances."[8] An impedance, usually employed in the frequency domain, is a ratio of voltage (or force) to current. As such, the term is only applicable to situations in which there is only one important independent variable (in this case longitudinal position or equivalently time). Since the centrifugal space-charge force described here has essential dependence also on both transverse coordinates it cannot be encompassed in the impedance formalism.

The force discussed in this paper has been neglected in all previous accelerator calculations. This has been shown to be a mistake, as its effects are large. It is not ruled out that this force is even dominant in some circumstances in the sense that beam-wall forces can be neglected without qualitative change in the phenomena. Such a claim could only be supported by further work, both theoretical and experimental.

I wish to acknowledge the contributions made by the CESR Operations Group. The data of Fig. 2 were taken with Lloyd Sakazaki using methods pioneered by Raphael Littauer and him. Some of the results have been checked by Glenn Decker. I have also profited from numerous conversations with Don Yennie and Andrew Sessler.

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