Invariance of the Spectrum of Light on Propagation

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The question is raised as to whether the normalized spectrum of light remains unchanged on propagation through free space. It is shown that for sources of a certain class that includes the usual thermal sources, the normalized spectrum will, in general, depend on the location of the observation point unless the degree of spectral coherence of the light across the source obeys a certain scaling law. Possible implications of the analysis for astrophysics are mentioned.

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Measurements of the spectrum of light are generally made some distance away from its sources and in many cases, as for example in astronomy, they are made exceedingly far away. It is taken for granted that the normalized spectral distribution of the light incident on a detector after propagation from the source through free space is the same as that of the light in the source region. I will refer to this assumption as the assumption of *invariance of the spectrum on propagation.* This assumption, which is implicit in all of spectroscopy, does not appear to have been previously questioned, probably because with light from traditional sources one has never encountered any problems with it. However, with the gradual development of rather unconventional light sources and with the relatively frequent discoveries of stellar objects of an unfamiliar kind, it is obviously desirable to understand whether all such sources generate light whose spectrum is invariant on propagation, and if so, what the reasons for it are. Actually it is not difficult to conceive of sources that generate light whose spectrum is not invariant on propagation. In this note I will show what are the characteristics of a certain class of sources that generate light whose spectrum is invariant, at least in the far zone.

From the standpoint of optical coherence theory, invariance of the spectrum of light on propagation from conventional sources is a rather remarkable fact, as can be seen from the following simple argument. Consider an optical field generated by a stationary source in free space. The basic field variable, say the electric field strength at the space-time point $(\mathbf{r},t)$, may be represented by its complex analytic signal[1,2] $E(\mathbf{r},t)$. According to the Wiener-Khintchine theorem[3] the spectral density of the light at the point $\mathbf{r}$ is then represented by the Fourier transform,

$$S(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\Gamma(\mathbf{r},\tau)e^{i\omega\tau}\,d\tau, \tag{1}$$

of the autocorrelation function (known in the optical context as the self-coherence function) of the field variable. It is defined as

$$\Gamma(\mathbf{r},\tau)=\langle E^*(\mathbf{r},t)E(\mathbf{r},t+\tau)\rangle, \tag{2}$$

where the angular brackets denote the ensemble average. Now the spectral density and the self-coherence function are the "diagonal elements" $(\mathbf{r}_2=\mathbf{r}_1=\mathbf{r})$ of two basic optical correlation functions, viz., the cross-spectral density

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\int_{-\infty}^{\infty}\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{i\omega\tau}\,d\tau, \tag{3}$$

and the mutual coherence function

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle E^*(\mathbf{r}_1,t)E(\mathbf{r}_2,t+\tau)\rangle. \tag{4}$$

It is well known that both the mutual coherence function and the cross-spectral density obey precise propagation laws. For example, in free space[4]

$$(\nabla_j^2+k^2)W(\mathbf{r}_1,\mathbf{r}_2,\omega)=0 \quad (j=1,2), \tag{5}$$

where

$$k=\omega/c, \tag{6}$$

with c being the speed of light *in vacuo* and ∇_j^2 being the Laplacian operator acting with respect to the variable $\mathbf{r}_j$. Consequently, both the mutual coherence function and the cross-spectral density and, in fact, also their normalized values change appreciably on propagation. For example, for a spatially incoherent planar source $W(\mathbf{r}_1,\mathbf{r}_2,\omega)$ and $\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)$ will be essentially δ correlated with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$ at the source plane but will have nonzero values for widely separated pairs of points which are sufficiently far away from the source. This is the essence of the well known van Cittert–Zernike theorem (Ref. 1, Sect. 10.4.2). In physical terms, the correlation in the field generated by a spatially incoherent source may be shown to have its origin in the process of superposition. We thus have the following rather strange situation: The correlations of the light may change drastically on propagation; yet, under commonly occurring circumstances, their (suitably normalized) diagonal elements, which represent the spectrum of the light or its Fourier transform, remain unchanged.

To obtain some insight into this problem we consider light generated by a very simple model source; namely, a planar source occupying a finite domain D of a plane $z=0$ and radiating into the half space $z>0$, which has the same spectral distribution $S^{(0)}(\omega)$ at each source point $P(\boldsymbol{\rho})$ and whose degree of spectral coherence[5] $\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$ is statistically homogeneous, i.e., has the functional form $\mu^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)$. The cross-spectral density of the light across the source plane is then given by

$$W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\epsilon(\boldsymbol{\rho}_1)\epsilon(\boldsymbol{\rho}_2)S^{(0)}(\omega)\mu^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega), \quad (7)$$

where $\epsilon(\boldsymbol{\rho})=1$ or 0 according to whether the point $P(\boldsymbol{\rho})$ is located within or outside the source area D in the plane $z=0$.

We will also assume that at each effective frequency ω present in the source spectrum, the linear dimensions of the source are much larger than the spectral correlation length [the effective width Δ of $|\mu^{(0)}(\boldsymbol{\rho}',\omega)|$]. Sources of this kind belong to the class of so-called *quasihomogeneous sources*,[6] which have been extensively studied in coherence theory in recent years. Most of the usual thermal sources are of this kind.

The radiant intensity $J_\omega(\mathbf{u})$, i.e., the rate at which energy is radiated at frequency ω per unit solid angle around a direction specified by a unit vector $\mathbf{u}$, is given by the expression [cf. Ref. 6, Eq. (4.8)]

$$J_\omega(\mathbf{u})=k^2AS^{(0)}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)\cos^2\theta. \quad (8)$$

In this formula, A is the area of the source,

$$\tilde{\mu}^{(0)}(\mathbf{f},\omega)=\frac{1}{(2\pi)^2}\int\mu^{(0)}(\rho',\omega)e^{-i\mathbf{f}\cdot\boldsymbol{\rho}'}d^2\rho' \quad (9)$$

is the two-dimensional spatial Fourier transform of the degree of spectral coherence, $\mathbf{u}_\perp$ is the transverse part of the unit vector $\mathbf{u}$, i.e., the component of $\mathbf{u}$ (considered as a two-dimensional vector) perpendicular to the z axis, and θ is the angle between the $\mathbf{u}$ and the z directions (see Fig. 1). Evidently the normalized spectral density $S^{(\infty)}(\mathbf{u},\omega)$ at a point in the far zone, in the direction specified by the unit vector $\mathbf{u}$, is given by

$$S^{(\infty)}(\mathbf{u},\omega)=J_\omega(u)/\int J_\omega(\mathbf{u})d\omega. \quad (10)$$

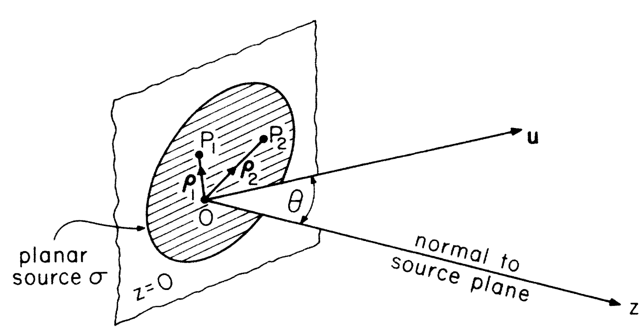

FIG. 1. Illustration of the notation.

On substituting Eq. (8) into Eq. (10) we obtain for the normalized spectrum in the far zone the expression

$$S^{(\infty)}(\mathbf{u},\omega)=\frac{k^2S^{(0)}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)}{\int k^2S^{(0)}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)d\omega}. \quad (11)$$

It is clear from Eq. (11) that the normalized spectrum of the light depends on the direction $\mathbf{u}$; i.e., it is in general not invariant throughout the far zone. However, it is seen at once from Eq. (11) that it will be invariant throughout the far zone if the Fourier transform of the degree of spectral coherence of the light in the source plane is the product of a function of frequency and a function of direction, i.e., it is of the form

$$\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)=F(\omega)\tilde{H}(\mathbf{u}_\perp). \quad (12)$$

In this case Eq. (11) reduces to

$$S^{(\infty)}(\mathbf{u},\omega)=\frac{k^2S^{(0)}(\omega)F(\omega)}{\int k^2S^{(0)}(\omega)F(\omega)d\omega}, \quad (13)$$

and the expression on the right is independent of the direction $\mathbf{u}$.

I will now show that the condition (12) has some interesting implications, which follow from the fact that $\mu^{(0)}$ is a correlation coefficient. Before doing this we note that since $\mathbf{u}$ is a unit vector, $|\mathbf{u}_\perp|<1$. However, we will now assume that the factorization condition (12) holds for all two-dimensional vectors $\mathbf{u}_\perp$ ($0\leq|\mathbf{u}_\perp|<\infty$). This assumption will be trivially satisfied if the degree of spectral coherence $\mu^{(0)}(\boldsymbol{\rho}',\omega)$ is, at each effective temporal frequency ω, band limited in the spatial frequency plane to a circle of radius k about the origin; in more physical terms this condition means that $\mu^{(0)}(\boldsymbol{\rho}',\omega)$ does not vary appreciably over distances of the order of the wavelength $\lambda=2\pi c/\omega$. With this being understood let us take the Fourier transform of Eq. (12). We then find at once that

$$\mu^{(0)}(\boldsymbol{\rho}',\omega)=F(\omega)\int\tilde{H}(\mathbf{u}_\perp)\exp(ik\mathbf{u}_\perp\cdot\boldsymbol{\rho}')\,d^2(k\mathbf{u}_\perp), \quad (14)$$

i.e.,

$$\mu^{(0)}(\boldsymbol{\rho}',\omega)=k^2F(\omega)H(k\rho'), \quad (15)$$

where H is, of course, the two-dimensional Fourier transform of $\tilde{H}$. Since $\mu^{(0)}(\boldsymbol{\rho}',\omega)$ is a correlation coefficient it has the value unity when $\rho'=0$, i.e.,

$$\mu^{(0)}(0,\omega)=1, \text{ for all } \omega, \quad (16)$$

and hence Eq. (15) implies that

$$k^2F(\omega)=[H(0)]^{-1}. \quad (17)$$

Since the left-hand side of Eq. (17) depends on the frequency but the right-hand side is independent of it, each side must be a constant (α say) and consequently

$$F(\omega)=\alpha/k^2. \tag{18}$$

Two important conclusions follow at once from these results. If we substitute Eq. (18) into Eq. (13) we obtain the following expression for the normalized spectrum of light in the far zone:

$$S^{(\infty)}(\mathbf{u},\omega)=S^{(\infty)}(\omega)=\frac{S^{(0)}(\omega)}{\int S^{(0)}(\omega)d\omega}. \tag{19}$$

This formula shows that not only is the normalized spectrum of the light now the same throughout the far zone, but it is also equal to the normalized spectrum of the light at each source point.

Next we substitute Eq. (18) into Eq. (15) and set $\alpha H=h$, $\rho'=\rho_2-\rho_1$. We then obtain for $\mu^{(0)}$ the expression

$$\mu^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)=h[k(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)] \quad (k=\omega/c); \tag{20}$$

i.e., the complex degree of spectral coherence is a function of the variable $\xi=k(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)$ only. We will refer to Eq. (20) as the *scaling law*. Obviously for a source that satisfies this law, the knowledge of the degree of spectral coherence of the light in the source plane at any particular frequency ω specifies it for all frequencies.

The scaling law (20), which ensures that for sources of the class that we are considering the normalized spectrum of the light is the same throughout the far zone and is equal to the normalized spectrum of the light at each source point [Eq. (19)], is the main result of this note.

It is natural to inquire whether sources are known that obey this scaling law. The answer is affirmative. Many of the commonly occurring sources, including blackbody sources, obey Lambert's radiation law [Ref. 1, Sect. 4.8.1]. It is known[7] that all quasi-homogeneous Lambertian sources have the same degree of spectral coherence, viz.

$$\mu^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)=\sin(k|\boldsymbol{\rho}_2-\boldsymbol{\rho}_1|)/k|\boldsymbol{\rho}_2-\boldsymbol{\rho}_1|, \tag{21}$$

which is seen to satisfy the scaling law (20). According to the preceding analysis such sources will generate light whose normalized spectrum is the same throughout the far zone and is equal to the normalized spectrum at each source point. This fact is undoubtedly largely responsible for the commonly held, but nevertheless incorrect, belief that spectral invariance is a general property of light.

This Letter has dealt with what is probably the simplest problem regarding spectral invariance on propagation. It would seem that some significant questions in this area might be profitably studied. Among them are the elucidation of the physical origin of the scaling law, spectral properties of light from a broader class of sources than considered here, the relation between the scaling law and Mandel's results regarding cross-spectrally pure light,[8,9] and relativistic effects. Applications of the results to problems of astrophysics might be of particular interest; at this stage one might only speculate whether source correlations may perhaps not give rise to differences between the spectrum of the emitted light and the spectrum of the detected light that originates in some stellar sources.

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$$\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\frac{W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)}{[W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_1,\omega)]^{1/2}[W^{(0)}(\boldsymbol{\rho}_2,\boldsymbol{\rho}_2,\omega)]^{1/2}}.$$

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