PHYSICAL REVIEW LETTERS

VOLUME 56 | 31 MARCH 1986 | NUMBER 13

Quantum Instability of de Sitter Space

I. Antoniadis[(a)]

Stanford Linear Accelerator Center, Stanford University, Stanford, California, 94305

J. Iliopoulos

Laboratoire de Physique Théorique, Ecole Normale Supérieure, Paris, France

and

T. N. Tomaras

Physics Department, University of Crete, Iraklion, Crete, Greece

(Received 28 October 1985)

The graviton propagator in a de Sitter background is found to be divergent. We show that as a consequence of this divergence, de Sitter space is not a solution of the equations of motion of the complete theory. If we start from de Sitter space as a classical ground state, quantum corrections change it into flat Minkowski space.

PACS numbers: 04.60.+n, 04.20.−q

An outstanding problem in the theory of gravitation is the observed vanishingly small value of the cosmological constant.[1] In perturbation theory this parameter can assume any value and one has to fine tune it to zero in order to have a vacuum state corresponding to flat Minkowski space. Several attempts have been made to argue that the zero value may be preferred dynamically[2,3] and in this Letter we want to discuss a new approach along this line. We shall present only our method and results and we shall leave all technical details for a lengthier publication.[4] Although all our formulas and conclusions can be trivially extended to d dimensions, in this Letter we shall restrict ourselves to $d=4$.

Our starting point is the observation that the propagator of a massless scalar field in four-dimensional de Sitter space is singular.[5] To be more precise, let us consider a scalar field described by

$$\mathscr{L}=\tfrac{1}{2}\sqrt{-g}\,[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)-m^2\phi^2]. \tag{1}$$

We choose the background metric of the form

$$g_{\mu\nu}=\bar{g}_{\mu\nu}=(t^2a^2)^{-1}\eta_{\mu\nu} \tag{2}$$

with $\eta_{\mu\nu}$ the flat Minkowski metric. The Lagrangean (1), after some algebra and a field rescaling, becomes $\mathscr{L}=\frac{1}{2}\phi D^{-1}\phi$ with

$$D^{-1}=-\partial_0^2-\mathbf{k}^2+t^{-2}(2-m^2/a^2). \tag{3}$$

The inverse of the operator D^{-1} depends on the initial conditions one chooses to impose, but for a choice which preserves the symmetries of de Sitter space,[5] D is given by

$$D(\mathbf{x},t;\mathbf{x}',t')=(tt')^{-1}(4\pi)^{-2}\Gamma(\tfrac{3}{2}-\nu)\Gamma(\tfrac{3}{2}+\nu)F(\tfrac{3}{2}-\nu,\tfrac{3}{2}+\nu,2;1+a^2\sigma^2/4), \tag{4}$$

where σ^2 is the invariant distance

$$\sigma^2=(a^2tt')^{-1}[(t-t')^2-(\mathbf{x}-\mathbf{x}')^2],$$

$\nu=(\frac{9}{4}-m^2/a^2)^{1/2}$, and F is a hypergeometric function. We see that for $m^2\rightarrow 0$ the propagator develops a pole of the form $1/m^2$.

Our argument will be based on the fact that the same pathology occurs also for the graviton propagator in de Sitter background. Let us first prove this statement. We start with the Lagrangean

$$\mathcal{L}=\sqrt{-g}\,[-(2/\kappa^2)R+\Lambda]. \tag{5}$$

We expand around the background given by Eq. (2) and we write $g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}$. The scalar curvature of the background is $\bar{R}=-12a^2=\Lambda\kappa^2$. We must now choose a gauge, isolate in (5) the quadratic part, and solve the corresponding equation of motion in order to find the graviton propagator. This is quite lengthy and tedious, and so we try to find a gauge in which the differential equation reduces to that of the scalar field whose properties have been studied in the literature. We found it convenient to have as much gauge freedom as possible and we promote the Lagrangean (5) into a locally conformally invariant one by introducing a scalar compensating field $\Phi(x)$:

$$\mathcal{L}=\sqrt{-g}\,[-\tfrac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu}-\tfrac{1}{12}R(V+\phi)^2+\lambda(V+\phi)^4], \tag{6}$$

where conformal invariance is spontaneously broken by the vacuum expectation value of $\Phi(x)$ which we wrote as $\phi(x)+V$ with $V=2\sqrt{6}/\kappa$. The coupling constant λ is equal to $\Lambda\kappa^4(24)^{-2}$. We wrote the Lagrangean (6) in four dimensions but it can be extended to arbitrary d, still preserving local conformal invariance.[6,7] Now we can impose five gauge conditions and we take them to be

$$h_{0\mu}=0,\quad h_i^i=0. \tag{7}$$

It is straightforward to compute the graviton propagator in this gauge. We must expand (6) around the background, and keep only terms quadratic in the fields. After some algebra, diagonalizing the $h-\phi$ terms and rescaling the fields we obtain $\mathcal{L}\sim\frac{1}{2}h^{ij}D_{ij,kl}^{-1}h^{kl}$ with

$$D_{ij,kl}^{-1}=\lim_{\xi\to 0}[(-\partial_0^2-\mathbf{k}^2+2/t^2)\eta_{ik}\eta_{jl}-2\mathbf{k}^2\eta_{ik}\omega_{jl}+\xi^{-1}\eta_{ij}\eta_{kl}], \tag{8}$$

where $\omega_{ij}=k_ik_j/\mathbf{k}^2$. The last term proportional to ξ^{-1} is due to the fifth gauge condition $h_i^i=0$. We can invert the operator (8) by introducing the complete set of three-dimensional projectors for symmetric rank-4 tensors $P^{(2)}$, $P^{(1)}$, $P^{(0-s\omega)}$, $P^{(0-\omega s)}$, $P^{(0-s)}$, $P^{(0-\omega)}$ whose precise definition is given by Antoniadis and Tsamis[7] and van Nieuwenhuizen.[8] We only notice that $P^{(2)}$ is the projector for the transverse and traceless spin-2 part. In terms of these operators we can show that

$$\begin{aligned}&[AP^{(2)}+BP^{(1)}+CP^{(0-s)}+D(P^{(0-s\omega)}+P^{(0-\omega s)}+EP^{(0-\omega)}]^{-1}\\&\quad=\frac{1}{A}P^{(2)}+\frac{1}{B}P^{(1)}-\frac{E}{D^2-CE}P^{(0-s)}+\frac{D}{D^2-CE}(P^{(0-s\omega)}+P^{(0-\omega s)})-\frac{C}{D^2-CE}P^{(0-\omega)}.\end{aligned} \tag{9}$$

In our case the operator (8) is of the form of the left-hand side of (9) with the identification

$$A=-\partial_0^2-\mathbf{k}^2+2/t^2,\quad B=A-\mathbf{k}^2,\quad C=A+2\xi^{-1},\quad D=\sqrt{2}\xi^{-1},\quad E=A-2\mathbf{k}^2+\xi^{-1}. \tag{10}$$

We see that A, whose inverse is precisely the coefficient of the transverse part of the propagator, equals the operator given by (3) with $m^2=0$.[9] It follows that at least this part contains the singularity of the massless scalar field.

We want to emphasize at this point that this result, which is valid in all dimensions, is not at all a feature of the special gauge we have chosen. The artificial conformal invariance was only introduced for convenience. We could start directly from the Lagrangean of Eq. (5), with no auxiliary Φ field. Here it is convenient to write $h_{\mu\nu}=H_{\mu\nu}+\frac{1}{4}\bar{g}_{\mu\nu}h$ and we use a covariant gauge $D^\mu H_{\mu\nu}=0$. In this gauge $H_{\mu\nu}$ is transverse and traceless. In terms of these fields the quadratic part of Eq. (5) is

$$\begin{aligned}\mathcal{L}_{\text{quad}}=\tfrac{1}{4}H_{\mu\nu}(-\Box+\tfrac{1}{6}\bar{R})H^{\mu\nu}\\-\tfrac{3}{32}h(-\Box-\tfrac{1}{3}\bar{R})h,\end{aligned} \tag{11}$$

where all derivatives are covariant with respect to the background. It is easy to see that the $H_{\mu\nu}$ propagator is nonsingular. The simplest way is to observe that the operator $-\Box+\frac{1}{6}\bar{R}$ applied on transverse and traceless second-rank tensors has no zero modes on S^4, which is the Euclidean version of de Sitter space.[10] On the other hand, the operator $-\Box-\frac{1}{3}\bar{R}$ for the scalar part h does[10] and this signals the presence of a singularity in the h propagator. Notice that, in this gauge, this is not an unphysical singularity. To see this, we can introduce a massive scalar matter field and compute the two-particle scattering amplitude. The singularity due to the one-h-exchange tree diagram does not cancel. Although a covariant gauge sounds simpler we prefer to continue our calculation in the gauge (7) for essentially two reasons: (i) In this gauge we are able to obtain the complete form of the graviton propagator, not just its singular part. (ii) It is only in the $h_{0\mu}=0$ gauge that the identification of the physical components is straightforward; they belong to the transverse and traceless part of h_{ij}. In the rest of this Letter we shall show that, as a result of the singular nature of the propagator, the ground state changes into flat Minkowski space.

An immediate consequence of our result is that a consistent quantization around a de Sitter background requires an infrared regularization. The simplest choice is a term linear in the quantum field h and so we add to the effective Lagrangean a term $\hat{r}h^{\mu}_{\mu}$, where $\hat{r}$ is a parameter with dimensions [mass]^4. This regularization breaks local coordinate as well as conformal invariance and gives a mass to the graviton. Consistency of the equations of motion requires the addition of a term $\hat{c}\Phi(x)$ with $\hat{c}$ equal to $8\hat{r}/V$. At the end the limit $\hat{r}\to 0$ must be taken.[11]

Let us now state our program: The theory, in its minimal form, contains a mass scale κ^{-1} or, more conveniently, V and two dimensionless parameters λ and $r-\hat{r}/V^4$. Every physical quantity can be computed as a function of them. We shall concentrate on the curvature R. It is obtained by setting equal to zero the coefficient of the linear term in h^{μ}_{μ} in the effective action. This coefficient is given by the sum of the graviton tadpole diagrams. In the tree approximation we find

$$R_{(0)}=24(\lambda+2r)V^2. \tag{12}$$

At the limit $r\to 0$ we obtain $\bar{R}=\Lambda\kappa^2$ as before. At higher orders we must introduce a suitable ultraviolet cutoff and, in practice, we shall use dimensional regularization. The result will be a function $R(V^2,\lambda,r,\epsilon)$, where $\epsilon=4-d$. The physical value is obtained by first taking the limit $r\to 0$ and then letting $\epsilon\to 0$.

Of course, we are not able to sum exactly the series of tadpole diagrams and obtain the exact value of R. Therefore, we must use some kind of approximation. In this Letter we shall present, as an illustration, the results of a first-order calculation with only one-loop diagrams taken into account. In order to make this truncation consistent we need an expansion scheme which makes one-loop diagrams of the same order as tree diagrams and all multiloop diagrams of higher order. This is the strategy followed by Coleman and Weinberg[12] for the study of massless scalar QED. For this we need two coupling constants, one much larger than the other, and the simplest way to introduce a second coupling constant is to add to the Lagrangean a term proportional to R^2. We could have considered the complete fourth-derivative theory, which contains one more term (for example, $R_{\mu\nu}R^{\mu\nu}$), but this only complicates the computation without adding any new feature. The conformal extension of a theory with R and R^2 terms can be written as

$$\mathscr{L}=\sqrt{-g}\left[\frac{1}{2\alpha^2}\left\{\frac{\Box\Phi}{\Phi}-\frac{R}{6}\right\}^2-\tfrac{1}{2}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)-\tfrac{1}{12}R\Phi^2+\lambda\Phi^4\right]+\hat{r}h^{\mu}_{\mu}+\hat{c}\Phi, \tag{13}$$

where α^2 is a new dimensionless coupling constant. We prefer to work with Lagrangeans containing only first derivatives and this can be easily achieved with the introduction of an auxiliary field $F(x)$.[3] After some rather lengthy algebra, we rewrite (13) in the classically equivalent form:

$$\begin{aligned}\mathscr{L}=\sqrt{-g}\Bigg[&\tfrac{1}{2}(\partial_\mu f)(\partial_\nu f)g^{\mu\nu}-\tfrac{1}{2}(\partial_\mu\sigma)(\partial_\nu\sigma)g^{\mu\nu}-\frac{R}{12}[(v+\sigma)^2-f^2]\\&-\frac{\alpha^2V^2}{2}(v_F+\sinh\omega\,\sigma+\cosh\omega\,f)^2\left(1+\frac{\sigma+f}{v}\right)^2+\lambda V^4\left(1+\frac{\sigma+f}{v}\right)^4\Bigg]+\hat{r}h^{\mu}_{\mu}+\hat{c}\frac{V}{v}(\sigma+f),\end{aligned} \tag{14}$$

where we have gone through the following steps of field redefinitions:

$$F=-\frac{1}{\alpha^2\Phi^2}\left(\Box\Phi-\frac{R}{6}\Phi\right),\quad \tilde{\Phi}=\Phi-F, \tag{15a}$$

$$F'=F-v_F,\quad \Phi'=\tilde{\Phi}-v_\Phi,\quad V=v_\Phi+v_F, \tag{15b}$$

$$\begin{pmatrix}\sigma\\ f\end{pmatrix}=\begin{pmatrix}\cosh\omega & -\sinh\omega\\ -\sinh\omega & \cosh\omega\end{pmatrix}\begin{pmatrix}\Phi'\\ F'\end{pmatrix},\quad \sinh\omega=\frac{v_F}{v},$$
$$\cosh\omega=\frac{v_\Phi}{v},\quad v=(v_\Phi^2-v_F^2)^{1/2}. \tag{15c}$$

The Lagrangean (14) is still not very convenient for calculations because it contains nondiagonal terms bilinear in h, σ, and f. In order to diagonalize the quadratic forms we introduce new fields:

$$\rho(x)=\sigma(x)+\tfrac{1}{3}f(x),$$
$$h'_{\mu\nu}=h_{\mu\nu}+2\bar{g}_{\mu\nu}\sigma/v, \tag{16}$$

in terms of which there are no mixed propagators. We are finally in a position to compute the curvature R, in terms of the independent parameters of our theory which we choose to be α^2, λ, and V. In the tree approximation, the vanishing of the coefficient of the linear term in h gives

$$-Rv^2-12\alpha^2V^2v_F^2+24\lambda V^4+48rV^4=0. \tag{17}$$

On the other hand, Eq. (15a) gives for v_F

$$v_F=R/6V\alpha^2. \tag{18}$$

The combination of (17) and (18) yields the value of $R_{(0)}$ given in Eq. (12). At higher orders we must compute all tadpole diagrams. The following remarks, results of a rather lengthy although straightforward calculation, simplify this task: (i) Only graviton internal lines give divergent contributions when r goes to zero. (ii) For the purposes of this computation, the three-graviton coupling constant turns out to be Rv^2 while each graviton propagator brings a factor $1/v^2$. (iii) For perturbation theory to be meaningful we must assume that $\alpha^2 << 1$. In fact, in this paper, we shall also assume that $\alpha^2 << \lambda$; in this case V^2/v^2 is of order α^2. With these remarks in mind, the one-loop corrections to Eq. (17) take the form

$$(A+B/r)R^2 - RV^2 + 24\lambda V^4 + 48rV^4 = 0, \qquad (19)$$

where we have omitted terms of order α^2. The term proportional to R^2 comes entirely from the one-loop calculation. We have split its contribution into two parts. The only diagram which contributes to the B term, the one which diverges when $r \to 0$, is the graviton loop. All others, scalar as well as Faddeev-Popov ghost loops, contribute only to A. Let us stress once more that (19), although extremely condensed, is just the result of an explicit calculation of all tree and one-loop contributions to the graviton tadpole. We see that, if we are allowed to truncate the loop expansion and solve (19) for R, we find, at the limit $r \to 0$, $R=0$. At this point the presence of the second coupling constant α^2 is crucial. Indeed, one can verify that the two-loop diagrams contribute to Eq. (19) terms of order $\alpha^2 R^2$ and similarly for higher loops with increasing powers of the coupling constant. It follows that, for $\alpha^2 << 1$, it is consistent to keep only the terms present in Eq. (21). We conclude that, although we started with a de Sitter background space-time, quantum corrections force the curvature to vanish. de Sitter space is not a solution of the equation of motion.

Before closing we want to remark that a completely nonperturbative proof should exist, similar in spirit to the ones showing the absence of spontaneous breaking of a continuous global symmetry in two dimensions,[13] establishing an inequality of the form $R \geqslant 0$, and thus excluding the case of de Sitter space.

We wish to thank S. Coleman, D. Z. Freedman, G. 't Hooft, B. Ratra, and E. Tomboulis for enlightening discussions. Two of us (J.I. and T.T.) are indebted, for the hospitality shown to them, to Professor S. D. Drell and the Theory Group at Stanford Linear Accelerator Center, where this work was initiated. I.A. thanks Professor C. Bouchiat and the Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, where the work was completed. For continued stimulation during the intervening time we all thank Professor G. Tiktopoulos and the Theory Group at the National Technical University of Athens. The work of one of us (I.A.) was supported in part by U.S. Department of Energy Contract No. DE-AC03-76SF00515. T.T. was supported in part by the Physics Research Institute of Crete. Laboratoire de Physique Théorique is a laboratoire propre du Centre National de la Recherche Scientifique associé à l'Ecole Normale Supérieure et à l'Université de Paris Sud.

---

[a]On leave of absence from Centre de Physique Théorique, Ecole Polytechnique, Palaiseau, France.

[1]See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, in *Gravitation* (Freeman, San Francisco, 1973), p. 410.

[2]S. W. Hawking, Phys. Lett. **134B**, 403 (1984); E. Mottola, Phys. Rev. D **31**, 754 (1985), and Institute of Theoretical Physics, University of California, Santa Barbara, Report No. NSF-ITP-84-159, 1984 (to be published).

[3]I. Antoniadis, J. Iliopoulos, and T. N. Tomaras, Nucl. Phys. **B261**, 157 (1985).

[4]I. Antoniadis, J. Iliopoulos, and T. N. Tomaras, to be published.

[5]L. H. Ford and L. Parker, Phys. Rev. D **16**, 245 (1977); A. D. Linde, Phys. Lett. **116B**, 335 (1982); S. W. Hawking and I. G. Moss, Nucl. Phys. **B224**, 180 (1983); S. W. Hawking, Nucl. Phys. **B239**, 257 (1984); B. Ratra, Phys. Rev. D **31**, 1931 (1935). For a general survey of quantum field theory in nontrivial background see N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, Cambridge, England, 1982), and references therein.

[6]F. Englert, C. Truffin, and R. Gastmans, Nucl. Phys. **B117**, 407 (1976).

[7]I. Antoniadis and N. Tsamis, Stanford Linear Accelerator Report No. SLAC-PUB-3297, 1984 (unpublished).

[8]P. van Nieuwenhuizen, Nucl. Phys. **B60**, 478 (1973).

[9]T. Banks, W. Fischler, and L. Susskind, Stanford Linear Accelerator Center Report No. SLAC-PUB-3367, 1984 (unpublished); L. H. Ford, Phys. Rev. D **31**, 710 (1985).

[10]G. W. Gibbons and M. J. Perry, Nucl. Phys. **B146**, 90 (1978); S. M. Christensen and M. J. Duff, Nucl. Phys. **B154**, 301 (1979), and **B170 [FS1]**, 480 (1980).

[11]A regulator of this kind is necessary even when quantizing around flat space, where the propagator is well defined, in order to regularize the graviton tadpole diagrams [see B. de Wit and R. Gastmans, Nucl. Phys. **B128**, 294 (1977) and I. Antoniadis, J. Iliopoulos, and T. N. Tomaras, Stanford Linear Accelerator Center Report No. SLAC-PUB-3768, 1985 (unpublished)]. However, in our case the role of the regulator is much more fundamental, because, in its absence, even the free propagator does not exist.

[12]S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).

[13]S. Coleman, Commun. Math. Phys. **31**, 259 (1973); S.-K. Ma and R. Rajaraman, Phys. Rev. D **11**, 1701 (1975).