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Improved Bounds on the Dimension of Space-Time

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We treat the perihelion shift of the planetary motion and the Lamb shift in hydrogen in an arbitrary number of space dimensions. Comparison with experimental data shows that the deviation from dimensionality four of space-time is less than 10^{-9} and 3.6×10^{-11}, respectively, on the length scales associated with these phenomena.

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In recent years the idea that the true number of dimensions of space-time may differ from four has attracted widespread attention. Based on the original suggestion of Kaluza[1] and Klein[2] various models have been developed with the aim of unifying all known interactions in a space of one timelike and D spacelike dimensions.[3,4] Of the latter all but three are thought to be compact with a radius of curvature so small that excitations of the additional degrees of freedom are possible only at ultrahigh energies, leaving only an effectively four-dimensional space-time at energies accessible to us. However, the mechanism responsible for fixing this effective number of dimensions to four is still a mystery (see, e.g., Terazawa[5] for possible solutions to this problem).

Furthermore, the difficulties associated with the incorporation of gravity into the structure of quantum field theory and, in general, the inherent infinities of the relativistic theory of quantum fields have inspired many physicists to consider the four-dimensional space-time continuum not as fundamental but as the low-energy appearance of a deeper theory. One of the most widely considered possibilities, that space-time on a small scale is a discrete lattice, is generally used today as a gauge-invariant regularization scheme in quantum field theory.[6] It is found that Lorentz invariance, broken microscopically by the lattice, is dynamically restored at large scales,[7] ruling out the main argument against this idea. More radical concepts altogether discard the notion of spatial points as fundamental entities and consider them, e.g., as expectation values of quantum variables associated with matter fields.[8]

If such considerations are anywhere near the truth, it is not *a priori* clear whether space-time, in the large, is a metric space of exactly four dimensions or whether its dimensionality differs, however slightly, from this integer value. (There is, in fact, no conceptual problem in considering physical phenomena in spaces with a noninteger number of dimensions, so-called fractals.[9] Arbitrary dimensionality has been widely used in the theory of phase transitions and renormalization theory, and it has been recently shown that fractal lattices can be applied for numerical calculations of Ising models in any number of dimensions.[10,11]) In this contribution we try to answer the question of what limits we can set with regard to the number of space dimensions, effectively observable at low energies, on the basis of existing experimental data.

A framework for an operational definition of noninteger dimensions of a metric space has recently been given by Zeilinger and Svozil[12] who generalized Hausdorff's concept[13] to account for finite experimental resolution δ. If we define the operational Hausdorff measure as

$$\mu_{\mathrm{op}}(\alpha,\delta) = \lim_{\epsilon \to \delta+} \left[\inf_{\{B_i\},\ \epsilon \geq \mathrm{diam} B_i \geq \delta} \sum_i (\mathrm{diam} B_i)^\alpha \right], \qquad (1)$$

where the $\{B_i\}$ represent all possible coverings of a region of space-time with diameters between δ and ϵ, the noninteger dimension $d(\delta)$ is defined, e.g., by

$$(\partial^2\mu_{op}/\partial\alpha^2)|_{\alpha=d}=0, \tag{2}$$

where d obviously may depend on the resolution δ (for details see Ref. 12). Zeilinger and Svozil pointed out that a value of $d<4$ would render all logarithmic divergences in quantum field theory finite. They also noted that the current discrepancy between theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of space-time is

$$d=4-(5.3\pm2.5)\times10^{-7}. \tag{3}$$

In this note we want to point out that other precision experiments set much more stringent limits on the deviation of d from the value 4. We shall assume in the following, in line with most other studies of this problem, that space-time has exactly one timelike and an arbitary number D of spacelike dimensions, i.e., $d=D+1$. The basic idea of our argument is to make use of the dynamical SO(4) invariance of motion in a $1/r$ potential.[14] If the number of spacelike dimensions D differs from 3, the Coulomb potential of a pointlike source falls off as r^{2-D} and the dynamical symmetry is broken.[15] This then leads to anomalous contributions to the Lamb shift in hydrogenic atoms and to the perihelion shift of planetary orbits. Note, however, that the physical effects of $D\neq3$ are, in general, different from those of a deviation from the $1/r$ potential caused by, e.g., a finite rest mass of the photon or graviton. In this case Gauss's law would be violated while, in our present considerations, we assume it to be valid exactly, albeit in $D\neq3$ dimensions. Experiments[16] designed to detect deviations from Gauss's law are, therefore, unable to provide a limit on the deviation of D from the value 3 under our assumptions.

Let us first analyze the problem of the perihelion shift in D space dimensions. The standard textbook treatment[17] of this problem is easily generalized. If we introduce the variables $x^\mu=(ct,r,\theta_1,\theta_2,\ldots,\theta_{D-2},\phi)$ with the metric

$$g_{\mu\nu}=\mathrm{diag}(g_{001},-g_{rr1}-r^2,-r^2\sin^2\theta_1,\ldots,-r^2\sin^2\theta_1\cdots\sin^2\theta_{D-2}) \tag{4}$$

the generalized Schwarzschild geometry outside of a spherically symmetric source is obtained by solving the $(D+1)$-dimensional Einstein equation. The result is

$$g^{rr}=g_{00}=1-\frac{2m}{r}\left(\frac{r_0}{r}\right)^{D-3}, \tag{5}$$

where r_0 is an arbitary length scale that may be absorbed in the definition of m. We treat r_0 as a purely phenomenological parameter assuming only that it is not too small ($>10^{-2000}$ m). Otherwise the logarithmic contributions would become important. The relation of the parameter r_0 to the microscopic structure of space remains to be investigated.

The geodesic equation for motion in the equatorial plane characterized by $\theta_i=\pi/2$ for $i=1,2,\ldots,D-2$ differs from those valid for $D=3$ only because of the modified form of g_{00} in Eq. (5). The final equation for the variable $u=1/r$ reads

$$d^2u/d\phi^2+u=Dmu^2(u/u_0)^{D-3}+(m/h^2)(D-2)(u/u_0)^{D-3}, \tag{6}$$

where $h=r^2\dot{\phi}=$const is the conserved angular momentum. This expression can be analytically continued to noninteger values of the dimension D.

As we are interested in small deviations from $D=3$, we expand the right-hand side of Eq. (6) around $D=3$, up to terms linear in $\epsilon=3-D$:

$$\frac{d^2u}{d\phi^2}+u\approx3mu^2\left(1-\frac{\epsilon}{3}-\epsilon\ln\frac{u}{u_0}\right)+\frac{m}{h^2}\left(1-\epsilon-\epsilon\ln\frac{u}{u_0}\right). \tag{7}$$

For nearly circular orbits we write

$$u(\phi)\approx A_0+A_1\cos\omega\phi \tag{8}$$

with $|A_1|<<A_0$, and by inserting into Eq. (7) we obtain

$$1-\omega^2=6mA_0-\epsilon m/A_0h^2\approx6m^2/h^2-\epsilon \tag{9}$$

with $A_0\approx m/h^2$. The perihelion shift per revolution is therefore

$$\Delta\phi=2\pi(\omega-1)$$
$$\approx-6\pi m^2/h^2+\pi\epsilon=\Delta\phi_0+\pi\epsilon, \tag{10}$$

where $\Delta\phi_0$ is the standard shift found in (3+1)-dimensional general relativity. As $\Delta\phi$ is known[18] to agree with $\Delta\phi_0$ to better than 5×10^{-3} for the planet Mercury, ϵ is bounded by

$$|\epsilon|<5\times10^{-3}|\Delta\phi_0/\pi|\approx10^{-9} \tag{11}$$

for astronomical length scales.

Let us next derive a bound for ϵ on microscopic scales by considering the corrections to the Lamb shift induced by $\epsilon\neq0$. To proceed we first generalize the usual treatment[19] of the level structure of hydrogenic atoms to arbitrary integer-dimensional space, $D = 3, 4, 5 \ldots$. The D-dimensional Schrödinger equation including relativistic and spin-orbit corrections is

$$\hat{H}\psi = (\hat{H}_0 + \hat{W}_1 + \hat{W}_2 + \hat{W}_3)\psi = E\psi \qquad (12)$$

with ($\hbar = c = 1$)

$$\hat{H}_0 = -\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{D-1}{r}\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2} + eA_0, \qquad (13a)$$

$$\hat{W}_1 = \frac{e\nabla^2 A_0}{8m^2}, \quad \hat{W}_2 = \frac{(E - eA_0)^2}{2m},$$
$$\hat{W}_3 = \frac{1}{2m^2 r}\frac{\partial(eA_0)}{\partial r}(\hat{\mathbf{s}}\cdot\hat{\mathbf{L}}). \qquad (13b)$$

For light atoms the terms $\hat{W}_i$, $i = 1, 2, 3$, can be treated as perturbations. A_0 is the solution of the D-dimensional Poisson equation. Whenever this does not lead to divergences we use a nuclear point charge yielding the Coulomb potential in D dimensions:

$$eA_0 = -(Z\alpha/r)(r_0/r)^{D-3}. \qquad (14)$$

However, for $\hat{W}_1$ we must use an extended charge distribution which modifies the potential inside the nuclear radius.

The eigenvalues of the operators $\hat{\mathbf{L}}^2$ (the quadratic Casimir invariant) and $\hat{\mathbf{s}}\cdot\hat{\mathbf{L}}$ for the generalized states of angular symmetry $s_{1/2}$ ($L = 0$, $j = \frac{1}{2}$), $p_{1/2}$ ($L = 1$, $j = \frac{1}{2}$), and $p_{3/2}$ ($L = 1$, $j = \frac{3}{2}$) can be determined by group-theoretical methods[20,21] as

$$\hat{\mathbf{L}}^2 = \begin{cases} 0 & (s_{1/2}), \\ D-1 & (p_{1/2}, p_{3/2}), \end{cases} \qquad (15)$$

$$2(\hat{\mathbf{s}}\cdot\hat{\mathbf{L}}) = \begin{cases} 0 & (s_{1/2}), \\ D-1 & (p_{1/2}), \\ 1 & (p_{3/2}). \end{cases} \qquad (16)$$

Having obtained the explicit dependence of the Hamiltonian on the number of dimensions for s and p states, we can analytically continue the energies to arbitrary values of D. Expanding around $D = 3$, we can calculate the coefficient of the term linear in $\epsilon = 3 - D$ with the help of the generalized Hellmann-Feynman theorem (which is derived using the D-dimensional normalization condition),

$$\left.\frac{dE_n}{dD}\right|_{D=3} = \int_0^\infty r^2\, dr\, f_3^{(n)}(r) \left.\frac{\partial\hat{H}}{\partial D}\right|_{D=3} f_3^{(n)}(r). \qquad (17)$$

By explicit calculation we find that the contributions to $E_{2s_{1/2}} - E_{2p_{1/2}}$ from the perturbations $\hat{W}_1$, $\hat{W}_2$, and $\hat{W}_3$ are small compared to that from $\hat{H}_0$ [smaller than $(10^{-2}\text{ eV})\epsilon Z^4$], if the length parameter r_0 in Eq. (14) is not very much smaller than the Planck length (10^{-33} cm). The D dependence of H_0 induces a splitting between the $2p_{1/2}$ and the $2s_{1/2}$ states even in the nonrelativistic limit which is linear in ϵ, because it exactly vanishes in three space dimensions as a result of the SO(4) symmetry of the quantum mechanical Kepler problem:

$$\Delta\tilde{E}_{\rm LS} = E_{3-\epsilon}^{(2p_{1/2})} - E_{3-\epsilon}^{(2s_{1/2})} \approx -\frac{(Z\alpha)^2 m}{12}\epsilon$$
$$\approx -Z^2\epsilon \times (2.27\text{ eV}). \qquad (18)$$

If the number of spatial dimensions should differ from 3, the contribution (18) would come in addition to the Lamb shift as calculated from radiative corrections of quantum electrodynamics. Since the latter increases as Z^4, the hydrogen atom provides the best experimental test for a possible nonzero value of ϵ. The present uncertainty of the experimental[22] and theoretical[23] values of the Lamb shift in hydrogen,

$$|\Delta E_{\rm LS}^{\rm exp} - \Delta E_{\rm LS}^{\rm th}| < 0.02\text{ MHz}$$
$$= 8.2\times 10^{-11}\text{ eV}, \qquad (19)$$

restricts the deviation from three dimensions to

$$|\epsilon| = |D - 3| < 3.6\times 10^{-11} \qquad (20)$$

on length scales comparable to the Bohr radius of hydrogen. This bound is more than 4 orders of magnitude smaller than the value (3) advocated in Ref. 12.

In conclusion, we have shown that the dynamical symmetry associated with motion in a $1/r$ potential provides extremely stringent limits on any possible deviation of the number of dimensions from the integer value of 3, on both atomic and astronomical length scales. In fact, the bound (20) makes D, besides the electron g factor, the best measured, but probably least understood, dimensionless constant in physics.

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Note added.—After completion of our manuscript we learned that Jarlskog and Ynduráin[24] have obtained the same limit $D - 3 \lesssim 10^{-9}$, by considering the effect on periastron motion.

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