Spin-Glass on a Bethe Lattice

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The Ising spin-glass in a magnetic field is studied for the Bethe lattice. There is an instability that agrees with the replica-symmetry–breaking transition found for the infinite-range model. Correlation lengths are finite on both sides of the transition, but there is a correlation length that diverges at the transition. Some features are different from those of the infinite-range model, and in particular the magnetic susceptibility and internal energy vary smoothly through the transition. An analogy with the localization transition on the Bethe lattice is pointed out.

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During the past ten years there have been many studies of the Sherrington-Kirkpatrick[1] model of a spin-glass. In this model each spin interacts with every other spin in the system, and it was reasonable to suppose that mean-field techniques would provide an exact solution of the model. While Sherrington and Kirkpatrick did indeed find an exact solution in the high-temperature (paramagnetic) phase by use of the "replica trick," which had been used for the spin-glass problem by Edwards and Anderson,[2] it was clear that their solution was wrong at low temperatures. Work by de Almeida and Thouless[3] showed that even in nonzero external magnetic field there was a transition at which the symmetry between replicas was broken, and that the failure of the Sherrington-Kirkpatrick solution at low temperatures was due to the neglect of replica-symmetry breaking, but no method of constructing the low-temperature phase was proposed in that paper. Parisi[4] found a way constructing a spin-glass phase with broken symmetry in which the Edwards-Anderson order parameter q is replaced by a continuous function $q(x)$ monotonic in the range $0 < x < 1$. The meaning of this continuous function was initially obscure. Sompolinsky[5] suggested that decreasing values of x should correspond to increasing time scales for a nonequilibrium system. The interpretation that now finds favor is that of De Dominicis and Young[6] and Parisi,[7] in which dx/dq is the probability density for an overlap of magnitude q between different possible equilibrium states of the spin-glass under given external conditions. There have been various discussions of the relation between the two approaches.[8] Recent work by Mézard *et al.*[9] has taken the approach of Refs. 6 and 7 further and has shown that the overlaps have an ultrametric property, so that out of three equilibrium states the two with most overlap have overlaps with the third state that are equal to one another.

Although the infinite-range model gives interesting clues about the behavior which might be expected for a finite-range spin-glass in a space of sufficiently high dimension, it has some properties that make it quite unlike a finite-range system. Because all sites are connected equally to one another there is no correlation length, there are no boundary conditions, and there is no way in which the limit of an infinite system can be approached by enlarging a finite system in such a way that finite neighborhoods are undisturbed. For this reason it seems valuable to reexamine the Bethe lattice as another system on which the spin-glass problem might be solved. It was pointed out by Kurata, Kikuchi, and Watari[10] that the first Bethe approximation provides an exact solution of the Ising model on an infinite Cayley tree, and Bowman and Levin[11] have shown how to obtain the solution for a spin-glass model, but they only studied the problem in the absence of a magnetic field. In this paper I examine the solution of the spin-glass problem on a Bethe lattice for small magnetic field in the neighborhood of the critical point. It is found that the transition occurs on a critical curve in the HT plane which is the same as that found for the infinite-range model by de Almeida and Thouless.[3] On either side of this critical curve the correlation functions fall off exponentially with distance, but on the curve there is one correlation length which diverges. There are some features that are different from those of the infinite-range model. Thermodynamic averages are smooth on the transition curve except in zero field, whereas there are cusps in the infinite-range model. The overlap functions behave quite differently.

There are actually some close analogies between this problem and the localization of electrons on a Bethe lattice.[12] In the localized regime the self-energy at a particular site is a unique function of the energy, but ceases to be so in the extended-states regime. This is parallel to the behavior of the effective field in this spin-glass model, which is locally determined in the paramagnetic phase but not in the spin-glass phase. Also the average Green's function is analytic at the localization transition, just as the thermodynamic averages are smooth at the spin-glass transition.

The Bethe lattice is an infinite regular Cayley tree, and it can be regarded as the limit of a finite tree if some care is taken only to consider properties of points in the interior. Each site is connected to $K+1$ neighbors, where $K > 1$. How this is done for the Ising model is described by Baxter.[13] A central site 0 is defined, and then for each site i one can calculate the effective field ξ_i it exerts on its neighbor further in towards the central site. This satisfies the equation

$$\xi_i = \tanh^{-1}\left[\tanh J_{ki} \tanh\left(h + \sum_j \xi_j\right)\right], \qquad (1)$$

where J_{ki} is the coupling of the site i to its inner neighbor k, h is the external field, and the sum is over the K outer neighbors of i. The factor of $1/k_B T$ has been absorbed into all these fields and coupling strengths. The effective field ξ is half the logarithm of the quantity x defined by Baxter. The magnetization at the central site is given by

$$M_0 = \tanh\left(h + \sum_i \xi_i\right), \qquad (2)$$

where the sum goes over its $K+1$ neighbors. At high temperatures (small values of J and h) this equation defines a convergent iterative process, so that the effective field, which depends only on those sites that are further out along a path from the central site through the site in question, depends primarily on the coupling strengths of bonds in the same neighborhood. In the spin-glass phase, however, the iterative process is not convergent, but, as was pointed out by Bowman and Levin,[11] if random fields are applied at the distant boundaries it can be used to define a convergent iterative process for the probability distribution of the effective field at each site. Thus the difference between the two phases is that in the paramagnetic phase the effective fields have well-defined values independent of the boundary conditions if the boundaries are sufficiently distant, while in the spin-glass phase the values of the effective fields are very sensitive to boundary conditions, and only a probability distribution can be determined without reference to boundary conditions. This sensitivity to boundary conditions can be taken as another manifestation of replica-symmetry breaking.

This system can be explored in detail where the fields are small, since a moment expansion can then be used; the various powers of the effective fields are averaged over a distribution of values of J. The method is illustrated here by taking zero external field and keeping fourth moments in the effective fields, but the results which are quoted were obtained by keeping all terms up to second order in the external field and up to sixth order in the effective fields. To leading order in these fields Eq. (1) becomes

$$\xi_i = \tanh J_{ki} \sum_j \xi_j - \tfrac{1}{3} \tanh J_{ki} \operatorname{sech}^2 J_{ki} \left(\sum_j \xi_j\right)^3. \qquad (3)$$

It follows from the structure of the iterative process that effective fields can only be correlated if they lie on the same path out from the central site, and it is assumed that the J_{ij} are symmetrically distributed about the origin, and so the effective fields have no odd moments. By taking powers of Eq. (3) and averaging the equations over the J the equations

$$\begin{aligned} \langle \xi^2 \rangle &= t_1 \langle \xi^2 \rangle - \tfrac{2}{3}(t_1 - t_2)[\langle \xi^4 \rangle + 3(K-1)\langle \xi^2 \rangle^2], \\ \langle \xi^4 \rangle &= t_2[\langle \xi^4 \rangle + 3(K-1)\langle \xi^2 \rangle^2] \end{aligned} \qquad (4)$$

are obtained, where

$$t_n = K\langle \tanh^{2n} J \rangle. \qquad (5)$$

For $t_1 < 1$ this has only the trivial solution with the moments zero, but for $t_1 > 1$ there is another solution,

$$\langle \xi^2 \rangle = \frac{t_1 - 1}{2(K-1)} \frac{1 - t_2}{t_1 - t_2}. \qquad (6)$$

It is clear that the relevant thermal variable in this model is $1 - t_1$, and $\langle \xi^2 \rangle$ is linear in this variable below the critical temperature.

This calculation does not distinguish between how much of the variance of ξ is due to the variation from sample to sample (or site to site) and how much is due to the variation at each site for a particular set of values of J. To find this another set of effective fields η is calculated for the same sample, but with different boundary conditions; I refer to this as a second replica. Multiplication of the equations for the two, possibly different, effective fields at the same site gives the equations

$$\langle \xi\eta \rangle = t_1 \langle \xi\eta \rangle - \tfrac{2}{3}(t_1 - t_2)[\langle \xi^3\eta \rangle + 3(K-1)\langle \xi^2 \rangle \langle \xi\eta \rangle], \quad \langle \xi^3\eta \rangle = t_2[\langle \xi^3\eta \rangle + 3(K-1)\langle \xi^2 \rangle \langle \xi\eta \rangle]. \qquad (7)$$

Comparison of this with Eq. (4) shows that there are two possible solutions, one in which $\langle \xi\eta \rangle$ is equal to $\langle \xi^2 \rangle$, which is what happens in the paramagnetic phase, and the other in which it is different and is equal to zero, which is what happens in the spin-glass phase in zero external field.

The stability of Eqs. (4) and (7) can be examined by study of the matrix which relates variations of these moments in one shell of sites at a given distance from the central site to variations in another shell. Differentiation of these equations yields the matrix equation

$$\begin{aligned} u_i &= [t_1 - 4(K-1)(t_1 - t_2)\langle \xi^2 \rangle] u_j - \tfrac{2}{3}(t_1 - t_2) v_j, \quad v_i = 6(K-1) t_2 \langle \xi^2 \rangle u_j + t_2 v_j, \\ w_i &= -2(K-1)(t_1 - t_2)\langle \xi\eta \rangle u_j + [t_1 - 2(K-1)(t_1 - t_2)\langle \xi^2 \rangle] w_j - \tfrac{2}{3}(t_1 - t_2) x_j, \\ x_i &= 3(K-1) t_2 \langle \xi\eta \rangle u_j + 3(K-1) t_2 \langle \xi^2 \rangle w_j + t_2 x_j, \end{aligned} \qquad (8)$$

where

$$u_i=\delta\langle\xi_i^2\rangle,\quad v_i=\delta\langle\xi_i^4\rangle,\quad w_i=\delta\langle\xi_i\eta_i\rangle,\quad x_i=\delta\langle\xi_i^3\eta_i\rangle. \tag{9}$$

This has two irrelevant eigenvalues close to t_2, which is of order K^{-1}. One of the two relevant eigenvalues is the one associated with fluctuations in $\langle\xi^2\rangle$, which to first order in $\langle\xi^2\rangle$ is

$$\lambda_1=t_1-4(K-1)\langle\xi^2\rangle. \tag{10}$$

Above the critical temperature this is t_1 and below it is $2-t_1$, and so it is always less than unity except at the critical point. To the same approximation the eigenvalue associated with fluctuations in $\langle\xi\eta\rangle$ is

$$\lambda_2=t_1-2(K-1)\langle\xi^2\rangle, \tag{11}$$

which is unity below the critical temperature. The correlation lengths can be defined as $-1/\ln\lambda$.

A consistent calculation up to sixth order in the effective fields and second order in the external field gives a more detailed picture of the spin-glass phase. There is always a solution of the generalization of Eqs. (7) and (4) in which $\langle\xi\eta\rangle=\langle\xi^2\rangle$, but below the critical point there may be a second solution of the form

$$\langle\xi\eta\rangle=-\frac{1}{2}\langle\xi^2\rangle+\left[\frac{1}{4}\langle\xi^2\rangle^2+\frac{3h^2(1-t_2)}{K(t_1-1)(K+Kt_2-2-t_2)}\right]^{1/2}, \tag{12}$$

provided that this gives $\langle\xi\eta\rangle<\langle\xi^2\rangle$. The condition that this solution exists is, as can be seen by comparison with Eq. (6),

$$h^2<(t_1-1)^3\frac{K(K+2Kt_2-2-t_2)}{6(K-1)^2(1-t_2)}, \tag{13}$$

which for large K agrees precisely with the replica-symmetry–breaking transition found by de Almeida and Thouless[3] apart from terms of order K^{-1}. The 8×8 matrix equation corresponding to Eq. (8) gives a modification of Eq. (11) which is

$$\lambda_2=1-\frac{h^2}{K\langle\xi^2\rangle}+\frac{K-1}{1-t_2}(K+2Kt_2-2-t_2)(2\langle\xi\eta\rangle^2-\tfrac{2}{3}\langle\xi^2\rangle^2). \tag{14}$$

This is unity on the critical curve (13), and is greater than unity below it for the paramagnetic solution, but for the spin-glass solution it is less than unity, so that the correlation length is finite except on the critical curve. The equations involving moments of ξ alone (single replica moments) have no singularity in nonzero field, and the associated correlation lengths are all finite.

The inclusion of moments of order $2n$ in this calculation leads to irrelevant eigenvalues close to t_n, and modification of the marginal eigenvalues by an amount of order $\langle\xi^{2n-2}\rangle$. Equations (10) and (14) give the only correlation lengths that diverge in the critical region, provided that the moments of the distributions characterize the distributions adequately.

This discussion shows that the spin-glass on a Bethe lattice has a replica-symmetry–breaking transition in nonzero magnetic field, and that the zero-field transition is a bicritical point at which the critical curve runs into the critical point for the Edwards-Anderson[2] transition. There are, however, features of the model which are different from those of the Parisi solution of the infinite-range model. The internal energy and magnetization depend only on moments of ξ, and so thermodynamic properties should behave smoothly through the transition; in particular, the nonlinear susceptibility should be smooth except in zero field. De Dominicis and Kondor[14] found a massless mode for the infinite-range model in the spin-glass state, but in this model the correlation lengths all seem to be finite except on the transition curve. It also seems, from preliminary studies, that the overlap probability distributions are roughly Gaussian in this model, whereas they have sharp features in the Parisi solution.[6,7]

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