Gauge Algebras in Anomalous Gauge-Field Theories

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We show that in an anomalous gauge-field theory Faddeev's Schwinger term can be removed by a particular renormalization of the Gauss-law operators and we construct a positive-definite Hamiltonian that commutes with these renormalized operators.

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It is generally believed that quantum field theories with anomalies are inconsistent. The principle of anomaly cancellation is one of model building's most important constraints and is the prime motivation of the current interest in ten-dimensional superstring theories as candidates for ultimate unification.[1] However, recently it has been suggested[2–5] that there might be a way to quantize anomalous chiral gauge theories consistently: Perhaps gauge invariance should be implemented as a second-class constraint[2–4] or maybe anomalies could be viewed as a mechanism for gauge-symmetry breaking.[5] If anomalous gauge theories can be quantized our present-day picture of physics would change drastically. This investigation should also provide new insight into the structure of anomaly-free quantum theories and could advance our understanding of important unresolved problems connected, for example, with dynamical symmetry breaking.

In the conventional approach to an anomalous quantum field theory the anomaly appears as a one-cocycle of the fermion determinant.[3,6] However, this cocycle can be removed.[3] Faddeev suggested[2] that as a consequence of the cocycle a Schwinger term should arise in the commutator of the generators of infinitesimal gauge transformations (Gauss's law) and his conjecture has been widely discussed.[4,7]

In this Letter we shall analyze gauge algebras in quantized chiral gauge theories. We show how Faddeev's two-cocycle can arise in the commutator of the generators of infinitesimal gauge transformations.[8] However, this cocycle can also be removed: We introduce new, renormalized generators that satisfy the Lie algebra of the gauge group. These generators commute with a renormalized quantum Hamiltonian and as a consequence gauge freedom can be eliminated with a first-class constraint. The Hamiltonian is positive and since the state vectors admit a positive-definite inner product the theory is unitary. However, only indirect arguments are found in support of both locality and Lorentz invariance.

We consider a single $(3+1)$-dimensional Weyl fermion minimally coupled to a non-Abelian gauge field in a complex representation of the gauge group. The quantum-mechanical configuration space is a Hilbert bundle with base $\mathscr{A}^3$, the space of all static gauge connections with $A_0=0$. The fiber is a tensor product of bosonic wave functionals and fermionic Fock states and serves as the representation space of the quantum Hamiltonian. In the $A_0=0$ gauge there is residual gauge freedom under time-independent gauge transformations and the elimination of this gauge freedom ensures the nontriviality of the fiber bundle.

Local sections are constructed by solving for the eigenstates of the single-particle Weyl Hamiltonian,

$$H[A]\langle \mathbf{x}|E\rangle = i\boldsymbol{\sigma}\cdot(\boldsymbol{\partial}+\mathbf{A})\langle \mathbf{x}|E\rangle = E\langle \mathbf{x}|E\rangle, \quad (1)$$

and second quantizing the fermionic field operators:

$$\Psi(\mathbf{x}) = \sum_E \langle \mathbf{x}|E\rangle a_E, \quad \Psi^\dagger(\mathbf{x}) = \sum_E \langle E|\mathbf{x}\rangle a_E^\dagger,$$

where $a_E, a_E^\dagger$ are the fermionic creation and annihilation operators. The Fock vacuum is then defined as the state with all negative-energy levels filled. However, since there is no canonical choice for its overall phase, the phase of all second-quantized Fock states $|F\rangle$ can be redefined by $\mathbf{x}$-independent but $A_i^a(\mathbf{x})$-dependent functionals $\chi(A)$, $|F\rangle \to \exp\{-i\chi(A)\}\times|F\rangle$. As a consequence we can associate with the Hilbert bundle a U(1) principal bundle which has a natural induced connection

$$\mathscr{A}_i^a(\mathbf{x}) = \langle \mathrm{vac},A|\frac{\delta}{\delta A_i^a(\mathbf{x})}|\mathrm{vac},A\rangle, \quad (2)$$

where $|\mathrm{vac},A\rangle$ is the Fock vacuum with background field $A_i^a(\mathbf{x})$. Upon parallel transport around a closed loop on $\mathscr{A}^3$ the Fock vacuum acquires a phase which is the integrated exponential of (2). This phase reflects the nontrivial holonomy of the Fock states on $\mathscr{A}^3$ and is also related to the one-cocycle in the fermion determinant.[9] From Ref. 9 we conclude that the U(1) curvature tensor corresponding to the connection (2) is given by

$$\mathscr{F}_{ij}^{ab}(\mathbf{x},\mathbf{y}) \overset{s\to 0}{=} \frac{1}{2}\sum_{E_1E_2}\langle E_1|\frac{\delta}{\delta A_i^a(\mathbf{x})}|E_2\rangle\langle E_2|\frac{\delta}{\delta A_j^b(\mathbf{y})}|E_1\rangle |E_1|^{-s}|E_2|^{-s}[\mathrm{sgn}(E_1)-\mathrm{sgn}(E_2)]. \quad (3)$$

Obviously F is a complicated nonlocal functional of A_i. In Ref. 9 we showed that it can be related to the η invariant of a *five*-dimensional Dirac operator, and in a local expansion in powers of A_i we found

$$\mathscr{F}_{ij}^{ab}(\mathbf{x},\mathbf{y}) = (i/12\pi^2) d^{abc} \epsilon^{ijk} A_k^c(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) + \dots . \tag{4}$$

This result was also derived by Jo[8] using diagrammatic techniques and the Bjorken-Johnson-Low limit. A direct computation shows that the regularization in (3) interferes with the Bianchi identity on $\mathscr{A}^3$: If the regularization is removed at an intermediate stage $\mathscr{F}$ will violate the Bianchi identity while a more careful treatment ensures the validity of this identity.

We now investigate how the nontriviality of (2) affects Gauss's law on the Hilbert bundle. For this we first evaluate the commutator of the following normal-ordered generators of infinitesimal gauge transformations:

$$G^a(\mathbf{x}) = D_i^{ab}(\mathbf{x})\, \delta/\delta A_i^b(\mathbf{x}) + \tfrac{1}{2}[\Psi^\dagger(\mathbf{x}), \lambda^a \Psi(\mathbf{x})], \tag{5}$$

where $D_i^{ab} = \delta^{ab}\partial_i + f^{abc} A_i^c$ is the gauge-covariant derivative.

It is obvious that the commutator of the bosonic operators is canonical,

$$\left[D_i^{ac}(\mathbf{x}) \frac{\delta}{\delta A_i^c(\mathbf{x})}, D_j^{bd}(\mathbf{y}) \frac{\delta}{\delta A_j^d(\mathbf{y})} \right] = f^{abc} D_k^{cd}(\mathbf{x}) \frac{\delta}{\delta A_k^d(\mathbf{x})} \delta(\mathbf{x}-\mathbf{y}). \tag{6}$$

In order to compute the remaining commutators we regulate the action of the functional derivative with respect to $A_i^a(\mathbf{x})$ on the single-particle kets as follows:

$$\frac{\delta}{\delta A_i^a(\mathbf{x})} |E\rangle \overset{s \to 0}{=} \sum_{E_1} |E_1\rangle \langle E_1| \frac{\delta}{\delta A_i^a(\mathbf{x})} |E\rangle |E_1|^{-s},$$

with obvious modifications for bras. We also need the regulated action of (3) on the fermionic operators, e.g.,

$$\frac{\delta}{\delta A_i^a(\mathbf{x})} a_E \overset{s \to 0}{=} -\sum_{E_1} \langle E| \frac{\delta}{\delta A_i^a(\mathbf{x})} |E_1\rangle a_{E_1} |E_1|^{-s}.$$

The regulated charge operator is

$$\tfrac{1}{2}[\Psi^\dagger(\mathbf{x}), \lambda^a \Psi(\mathbf{x})] = \rho^a(\mathbf{x}) \overset{s \to 0}{=} \frac{1}{2} \sum_{E_1, E_2} \langle E_1|\mathbf{x}\rangle \lambda^a \langle \mathbf{x}|E_2\rangle [a_{E_1}^\dagger, a_{E_2}] |E_1|^{-s} |E_2|^{-s},$$

and its expectation value in the Fock vacuum is

$$\langle \mathrm{vac}; A| \rho^a(\mathbf{x}) |\mathrm{vac}; A\rangle = -\tfrac{1}{2}\eta^a(\mathbf{x}) \overset{s \to 0}{=} -\frac{1}{2} \sum_E \langle E|\mathbf{x}\rangle \lambda^a \langle \mathbf{x}|E\rangle |E|^{-s} \operatorname{sgn}(E). \tag{7}$$

A direct computation gives

$$[D_i^{ac}(\mathbf{x})\, \delta/\delta A_i^c(\mathbf{x}), \rho^b(\mathbf{y})] + [\rho^a(\mathbf{x}), D_i^{bc}(\mathbf{y})\, \delta/\delta A_i^c(\mathbf{y})] = 0, \tag{8}$$

$$[\rho^a(\mathbf{x}), \rho_b(\mathbf{y})] = f^{abc} \rho^c(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) - \{ D_i^{ac}(\mathbf{x}) D_j^{bd}(\mathbf{y}) \mathscr{F}_{ij}^{cd}(\mathbf{x},\mathbf{y}) - \tfrac{1}{2} f^{abc} \eta^c(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) \}, \tag{9}$$

and combining Eqs. (6), (8), and (9) we find for the commutator of (5)

$$[G^a(\mathbf{x}), G^b(\mathbf{y})] = f^{abc} G^c(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) - \{ D_i^{ac}(\mathbf{x}) D_j^{bd}(\mathbf{y}) \mathscr{F}_{ij}^{cd}(\mathbf{x},\mathbf{y}) - \tfrac{1}{2} f^{abc} \eta^c(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) \}. \tag{10}$$

It is straightforward to verify that the Schwinger term in (10) is a two-cocycle, i.e., it satisfies

$$D_i^{ad}(\mathbf{x})\, \delta \mathscr{S}^{bc}(\mathbf{y},\mathbf{z})/\delta A_i^d(\mathbf{x}) + f^{bcd} \mathscr{S}^{ad}(\mathbf{x},\mathbf{y}) \delta(\mathbf{y}-\mathbf{z}) + (\text{cyclic}) = 0. \tag{11}$$

A direct computation of the Schwinger term $\mathscr{S}^{ab}(\mathbf{x},\mathbf{y})$ is very complicated since both $\mathscr{F}$ and η^c are nonlocal functionals of A_i. However, from (3) and (7) we conclude that if $\mathbf{x} \neq \mathbf{y}$ this Schwinger term vanishes. Consequently $\mathscr{S}^{ab}(\mathbf{x},\mathbf{y})$ must have a local expansion in $\delta(\mathbf{x}-\mathbf{y})$ and its derivatives. On dimensional grounds this expansion truncates after $O(A^3)$, and if we use (4) and (11) to expand $\mathscr{S}^{ab}(\mathbf{x},\mathbf{y})$ in powers of A_i we find that (10) agrees with Faddeev's conjecture.

Since the generator (5) involves a composite operator it must be renormalized and consequently there is some freedom in its definition. In an anomaly-free theory we can modify (5) provided the redefined generators also satisfy the Lie algebra of the gauge group. However, for an anomalous theory the Lie-algebra constraint cannot be implemented *a priori:* For these theories we have arrived at the extended algebra (10) and thus we expect that now there is more latitude in the renormalization of (5). Instead of $G^a(\mathbf{x})$ we may consider, e.g.,

$$T^a(\mathbf{x}) = D_i^{ab}(\mathbf{x})\{\delta/\delta A_i^b(\mathbf{x}) - \mathscr{A}_i^b(\mathbf{x})\} + \tfrac{1}{2}[\Psi^\dagger(\mathbf{x}), \lambda^a\Psi(\mathbf{x})] + \tfrac{1}{2}\eta^a(\mathbf{x}), \tag{12}$$

which differs from (5) only by a vacuum subtraction. We find that these renormalized operators *satisfy* the Lie algebra of the gauge group,

$$[T^a(\mathbf{x}), T^b(\mathbf{y})] = f^{abc}T^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}).$$

Residual gauge freedom can now be eliminated by the first-class constraint

$$T^a(\mathbf{x})|\text{phys}\rangle = 0. \tag{13}$$

Similarly, the Hamiltonian also involves composite operators and must be renormalized. Since

$$[T^a(\mathbf{x}), \delta/\delta A_i^b(\mathbf{y}) - \mathscr{A}_i^b(\mathbf{y})] = f^{abc}[\delta/\delta A_i^c(\mathbf{x}) - \mathscr{A}_i^c(\mathbf{x})]\delta(\mathbf{x}-\mathbf{y}),$$

we conclude that the *simplest* modification of the *naive* Hamiltonian which commutes with $T^a(\mathbf{x})$ is

$$\mathscr{H}[A] = -\frac{1}{2}\int d^3x\left\{\frac{\delta}{\delta A_i^a(\mathbf{x})} - \mathscr{A}_i^a(\mathbf{x})\right\}^2 + \frac{1}{2}\int d^3x\, B_i^a(\mathbf{x})^2 + \frac{1}{2}\sum_E E[a_E^\dagger, a_E] + \tfrac{1}{2}\zeta_H(1), \tag{14}$$

where $\zeta_H(1)$ refers to the ζ function of (1) and ensures the positivity of (14). Since the constraint defines a subspace with a positive-definite inner product, by the Hermiticity of (14) the quantum theory (13),(14) is unitary. However, since the U(1) gauge field $\mathscr{A}_i$ is a complicated functional of A_i it is possible that (14) does not describe a Lorentz-invariant dynamics. In order to investigate whether a manifestly local and Lorentz-invariant formulation exists we now consider a candidate for the functional-integral representation of the theory.

We first recall that the second-quantized fermionic states do not admit a canonical choice for their overall phase. This phase can always be redefined by an $\mathbf{x}$-independent but A_i-dependent functional $\chi[A]$, corresponding to a static canonical transformation. As a consequence the *naive* transition amplitudes cannot be invariant under such a transformation and the relative [$A_i^a(\mathbf{x})$-dependent] phase of the fermionic Fock vacua $|\text{out}, T\rangle$ and $|\text{in}, -T\rangle$ with time $T \to \infty$ is arbitrary. Thus the matrix element $\langle \text{out}, T|\text{in}, -T\rangle$ cannot be defined; but if we introduce

$$\langle \text{out}, T|\text{in}, -T\rangle \to \langle \text{out}, T|\text{in}, -T\rangle \exp\left\{i\int_{-T}^{T} dA_i^a(t)\mathscr{A}_i^a(t)\right\}, \tag{15}$$

the matrix element becomes invariant under a redefinition of the phases. From Ref. 9 we also conclude that the additional term in (15) cancels the gauge dependence of the fermion determinant, which implies that the effective Lagrangean that appears in the functional-integral representation must be different from the classical Lagrangean, the additional counterterm canceling the gauge noninvariance of the fermionic measure. The results of Ref. 9 suggest that the counterterm is essentially the η invariant of a *five*-dimensional Dirac operator and, modulo trivial redefinitions, it can be identified with the Wess-Zumino Lagrangean L_{WZ}.[10] The functional integral for an anomalous theory is then[4]

$$\text{Tr}\{\exp(-i\mathscr{H}[A]T)\} = \int [dA][d\bar\psi][d\psi]\exp\left\{i\int_{-T/2}^{T/2} dx\left(-\tfrac{1}{4}F^2 + \bar\psi \boldsymbol{D}\psi + L_{\text{WZ}}\right)\right\}, \tag{16}$$

which is gauge invariant since L_{WZ} cancels the gauge noninvariance of the fermion determinant. We emphasize that the counterterm has a dynamical origin: The trace on the left-hand side is evaluated over states that are subject to the constraint (13) and L_{WZ} reflects the nontriviality of the subtraction introduced in (12).

If we introduce nondynamical scalar ghost fields the conventional Wess-Zumino term[10] is manifestly local and Lorentz invariant. Hence (16) must define a local and Lorentz-invariant quantum theory. Since the Wess-Zumino Lagrangean exhibits parameter quantization (16) should also be renormalizable.[4] However, since we have not been able to find a first-principles derivation of (16) starting from the constraint (13) and the proper Hamiltonian, we have not been able to verify explicitly whether locality and Lorentz invariance, which are manifest in (16), are compatible with unitarity and positivity, which are manifest in the Schrödinger formulation.

In conclusion, we have analyzed the gauge algebra in anomalous chiral gauge theories and found that it is possible to quantize these theories in a way that is manifestly gauge invariant, unitary, and positive. We have also argued that the effective Lagrangean in the path integrals is different from the classical Lagrangean. The additional term has its origin in the nontriviality of the quantum holonomy and general arguments imply that it can be identified with the Wess-Zumino Lagrangean. However, since we have not been able to show the equivalence of (13) and (14) [or a suitable modification of (14)] with (16), we cannot conclude that unitarity and positivity are compatible with locality and Lorentz invariance. This question is the subject of ongoing investigation.

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