Quantum Holonomy and the Chiral Gauge Anomaly

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(Received 15 July 1985)

The chiral gauge anomaly is studied by use of the U(1) holonomy on the space of all gauge fields. On this space an Abelian gauge structure is identified and the anomaly is related to the pertinent field-strength tensor and computed by use of the η invariant of a five-dimensional Dirac operator.

PACS numbers: 11.30.Rd, 11.10.Ef, 11.15.Tk

The analysis of chiral anomalies has led to impressive progress in our understanding of the nonperturbative aspects of quantum theories.[1] Anomaly cancellation is an important constraint on model building[2] and has played a central role in the identification of certain ten-dimensional string theories as strong candidates for ultimate unification.[3] Recent advances are largely based on the analysis of a particular solution[4] to the Wess-Zumino consistency condition[5] by use of group theory, cohomology,[6] and family index theorems.[7] However, aside from diagrammmatic computations[8] there has been little progress in the understanding of the origin of the chiral anomaly in terms of the quantum structure of the underlying field theory: It would be desirable to find an interpretation using such simple quantum mechanical concepts as the connection between the axial U(1) anomaly, spectral flow, and particle production.[9]

In this Letter we shall examine the quantum mechanical adiabatic approximation as a powerful nonperturbative technique for analyzing quantum field theories. As an example we consider the chiral gauge anomaly and present a first-principles computation of the effective action for the anomaly. We identify the Fock vacuum expectation value of the non-Abelian electric field operator as an induced U(1) connection in the infinite-dimensional space of all gauge fields and analyze the U(1) gauge structure in this space.[10] We find that the effective action for the anomaly is a surface integral of the pertinent U(1) field-strength tensor. We then proceed with the semiclassical quantization of the gauge fields as an infinite-dimensional quantum mechanical system of a point particle in an external "magnetic" gauge field, and show that the commutator algebra of the electric field operator forms a nonassociative Malcev algebra.[8,11] More details and extensions of our approach will be reported elsewhere.

Consider a single, minimally coupled Weyl fermion in a *complex* representation of some non-Abelian gauge group. We fix $A_0=0$ and consider the infinite-dimensional manifold $\mathscr{A}^3$ of all static gauge field configurations $A_i^a(\mathbf{x})$. On $\mathscr{A}^3$ a time-dependent gauge field $A_i^a(\mathbf{x},t)$ corresponds to a path and a periodic gauge field to a closed loop. In a nonanomalous theory we know how Gauss's law can be used to eliminate the residual static gauge freedom. However, since we consider an anomalous theory we do not attempt to project $\mathscr{A}^3$ into $\mathscr{A}^3/\mathscr{G}^3$, the space of three-dimensional gauge fields modulo three-dimensional gauge transformations. For present purposes it is enough to consider $\mathscr{A}^3$. On the periodic family of gauge fields $A_i^a(\mathbf{x},t)$ $(0\leq t<T)$ we consider the $T\to\infty$ limit of

$$\int D\psi^\dagger D\psi \exp\left\{i\int_0^T dt\,\psi^\dagger[i\partial_t+H(t)]\psi\right\}=\prod_n \lambda_n. \quad (1)$$

Here λ_n are the eigenvalues of the Dirac operator

$$\{i\partial_t+H(t)\}\psi(\mathbf{x},t)=\lambda\psi(\mathbf{x},t) \quad (2)$$

subject to $\psi(T)=-\psi(0)$ and $H(t)$ is the Dirac Hamiltonian. It depends on t through the background gauge fields and $H(T)=H(0)$. The zero modes of (2), $f_r(\mathbf{x},t)$ with $f_r(\mathbf{x},T)=\exp\{-i\alpha_r\}f(\mathbf{x},0)$, are characterized by the Floquet indices α_r,[12] and all eigenmodes of (2) are obtained from

$$\psi_{n,r}(\mathbf{x},t)=e^{-i(\omega_n-\alpha_r/T)t}f_r(\mathbf{x},t) \quad (3)$$

with $\omega_n=(2n+1)\pi/T$. Substituting (3) into (2) we get $\lambda_{n,r}=\omega_n-\alpha_r$ and a direct computation[12] yields for (1)

$$\exp(\tfrac{1}{2}i\textstyle\sum_r|\alpha_r|)\prod_r(1+e^{-i|\alpha_r|}). \quad (4)$$

For large T and with the appropriate Feynman boundary condition, we identify the first factor as the vacuum persistence amplitude. We compute (4) in the limit $T\to\infty$ using adiabatic approximation, and for simplicity we assume that for all t the eigenstates of $H(t)$, $H(t)\langle\mathbf{x}|r;t\rangle=E_r(t)\langle\mathbf{x}|r;t\rangle$, are nondegenerate and also that as a function of t the eigenvalues $E_r(t)$ do not cross zero, i.e., that there is no spectral flow.[9] Instead of $A_i^a(\mathbf{x})$ we find it convenient to use the notation A_x where x stands for a, i, $\mathbf{x}$, and all repeated indices are summed. With this notation the zero modes of (2) are given in the adiabatic approximation by

$$f_r(\mathbf{x},t)=\exp\left[i\int_0^t d\tau\,E_r(\tau)+i\gamma_r(t)\right]\langle\mathbf{x}|r;t\rangle, \quad (5)$$

$$\gamma_r(t)=i\int_0^t d\tau\langle r;\tau|\frac{d}{d\tau}|r;\tau\rangle$$
$$=i\int_0^t d\tau\frac{d}{d\tau}A_x(\tau)\langle r;\tau|\frac{\delta}{\delta A_x(\tau)}|r;\tau\rangle.$$

For an open path on $\mathscr{A}^3$ we can choose $\gamma_r(t)=0$. However, as pointed out by Berry,[13] for a closed loop ∂C on $\mathscr{A}^3$ this is not always possible: A nontrivial

$\gamma_r(T)$ implies nontrivial U(1) holonomy on A^3.[14] From (5) we then determine the Floquet index as $\alpha_r = -\int_0^T d\tau\, E_r(\tau) - \gamma_r(\partial C)$. Using the boundary condition that for time-independent background fields $\alpha_r = -E_r T$,[12] we conclude that $\gamma_r(\partial C)$ is $O(1)$ while the integral over the energy is $O(T)$. In the limit $T \to \infty$, (4) then reduces to

$$\exp\left\{\tfrac{1}{2} i \sum_r \int_0^T d\tau |E_r(\tau)| + \tfrac{1}{2} i \sum_r \operatorname{sgn}(E_r)\gamma_r(\partial C)\right\}. \tag{6}$$

(These infinite sums can be defined by use of a ζ-function regularization.) The first sum in (6) cannot contribute to the chiral anomaly: Upon the addition of an appropriate anomaly-cancelling Weyl fermion its exponent will be multiplied by 2. (It can therefore be considered as $\frac{1}{2}$ of a similar term in a regularized anomaly-free theory.) On the other hand, the second sum does not have any obvious CP symmetry and consequently it must contain the anomaly. In the following only this term will be considered. We appeal to the fixed-time Schrödinger formalism and second quantize the fermions in the time-dependent background of the gauge fields.[15] On the fermionic states the functional derivative operator then acts as the bilinear

$$\frac{\delta}{\delta A_x(\tau)} = \frac{1}{2} \sum_{r,r'} \langle r,\tau | \frac{\delta}{\delta A_x(\tau)} | r';\tau\rangle [a_r^\dagger, a_{r'}],$$

where the creation and annihilation operators $a_r^\dagger$ and a_r depend on τ through $A_x(\tau)$. The Fock vacuum $|\text{vac};\tau\rangle$ is the state with all positive-energy levels empty and negative-energy levels occupied and

$$\langle \text{vac};\tau | \frac{\delta}{\delta A_x(\tau)} |\text{vac};\tau\rangle = \frac{1}{2} \sum_r \operatorname{sgn}(E_r) \langle r;\tau | \frac{\delta}{\delta A_x(\tau)} | r;\tau\rangle = \mathscr{A}_x(\tau), \tag{7}$$

where $\mathscr{A}_x$ can be interpreted as a U(1) gauge field with base $\mathscr{A}^3$. (It can also be viewed as the non-Abelian electric field induced by vacuum polarization.) Indeed, the matrix element (7) is ambiguous: We can redefine the phase of the fermionic wave functions (and thereby the Fock vacuum) by any A-dependent but $\mathbf{x}$-independent functional $F[A]$. This corresponds to the U(1) gauge transformation $\mathscr{A} \to \mathscr{A} + dF[A]$, where $\mathscr{A} = \mathbf{d}_x A_x$ and $\mathbf{d} = dA_x\, \delta/\delta A_x$ is the exterior derivative on $\mathscr{A}^3$. Substituting (7) into (6) we find

$$\frac{1}{2} \sum_r \operatorname{sgn}(E_r)\gamma_r(\partial C) = -\int_0^T d\tau\, \partial_\tau A_x(\tau) \langle \text{vac};\tau | \frac{\delta}{\delta A_x(\tau)} |\text{vac};\tau\rangle = -\oint_{\partial C} dA_x \mathscr{A}_x. \tag{8}$$

We observe that this term gives the leading semiclassical correction to the kinetic part of the classical action. Furthermore, (8) is the Bohr-Sommerfeld integral $\oint dq_n\, p_n$ which would equal $2\pi n$ for a stationary state. Obviously this can only be achieved if there is no chiral gauge anomaly. Using Stokes's theorem we convert the line integral (8) into an integral over a surface C bounded by the loop ∂C,

$$\oint_{\partial C} dA_x \mathscr{A}_x = \int_C \mathscr{F}, \tag{9}$$

where $\mathscr{F} = \mathbf{d}\mathscr{A}$ is the field-strength two-form. If we integrate $\mathscr{F}$ over a topological two-sphere S^2 in $\mathscr{A}^3$ the surface independence of (9) yields $\oint F = 2\pi n$. If n vanishes for *every* two-surface on $\mathscr{A}^3$ the field strength $\mathscr{F}$ is exact, i.e., we can find a *global* $\mathscr{A}$ such that $\mathscr{F} = \mathbf{d}\mathscr{A}$ on $\mathscr{A}^3$. The loop integral (9) then reduces to an integral of the original gauge field $A_i^a(\mathbf{x},t)$ over the physical four-space and its divergences can be canceled by four-dimensional counterterms. A chiral gauge anomaly can thus arise *only* if $\mathscr{F}$ is not exact. This means that the relation $\mathscr{F} = \mathbf{d}\mathscr{A}$ can be valid at most locally on $\mathscr{A}^3$. But if we could find a region in $\mathscr{A}^3$ such that $\mathscr{F} = \mathbf{d}\mathscr{A}$ for some $\mathscr{A}$, then for every loop in this region the integral (9) could be represented by a four-dimensional integral over the physical space. Since the anomaly functional must be "analytic" in $A_i^a(\mathbf{x})$ the anomaly could then be canceled by a counterterm obtained by analytic continuation in $\mathscr{A}^3$. Hence a chiral gauge anomaly can only arise if the Bianchi identity for $\mathscr{F}$ fails, i.e., $\mathbf{d}\mathscr{F} \neq 0$ almost everywhere, which is possible only if $\mathscr{F}$ has a source with dense support on $\mathscr{A}^3$.[16] We shall now compute $\mathscr{F}$: We first show that (modulo gauge-invariant terms) $\mathscr{F}$ is related to the η invariant (i.e., to the number of positive eigenvalues minus the number of negative eigenvalues) of the *five*-dimensional Euclidean Dirac operator

$$D = i\Gamma^5 \partial_s + i\Gamma^4 \partial_t + i\Gamma^i[\partial_i + A_i(s)] = i\Gamma^5 \partial_s + Q, \tag{10}$$

defined in the cylinder $[-\infty,\infty] \times S^4$. Here Γ^μ are the $d=5$ Euclidean Dirac matrices and the s-dependent background gauge field $A_i(\mathbf{x},t;s)$ interpolates adiabatically between $A_i = 0$ at $s = -\infty$ and the original field $A_i(\mathbf{x},t)$ at $s = \infty$. For simplicity we assume that the eigenfunctions are nondegenerate and that there is no spectral flow. We consider the following representation of the η invariant[17]:

$$\eta\{D\} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega\, \mathrm{Tr}\left\{\frac{D}{D^2 + \omega^2}\right\}.$$

Expanding the denominator up to first order in derivatives of s we find, after some algebra,

$$\eta\{D\} = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, \mathrm{Tr}\left\{Q\frac{1}{-\partial_s^2+Q^2+\omega^2}i\partial_s Q\Gamma^5\frac{1}{-\partial_s^2+Q^2+\omega^2}\right\}.$$

Introducing the (s-dependent) eigenvalues $\lambda_n(s)$ and the corresponding eigenstates $|n;s\rangle$ of the Hermitean operator Q, we then find in the leading order of s derivatives

$$\eta\{D\} = \frac{1}{2\pi}\sum_n \int_{-\infty}^{\infty} ds \frac{1}{\lambda_n(s)}\langle n;s|i\partial_s Q\Gamma^5|n;s\rangle. \tag{11}$$

Since we are only interested in that part of $\mathscr{F}$ that gives rise to the chiral gauge anomaly, it is enough to consider only the parity-odd part of the Euclidean effective action, $I(A_i) = \frac{1}{2}\mathrm{Tr}\ln\{Q\Gamma^5\}$. If we again introduce the interpolation $A_i(s)$ a simple computation shows that up to a field-independent constant

$$I(A_i) = \tfrac{1}{2}\int_{-\infty}^{\infty} ds \frac{d}{ds} I(A_i(s)) = \tfrac{1}{2}\int ds\, \mathrm{Tr}\left\{\frac{1}{Q}\partial_s Q\Gamma^5\right\} = \tfrac{1}{2}\sum_n \int ds \frac{1}{\lambda_n(s)}\langle n|\partial_s Q\Gamma^5|n\rangle.$$

Comparing with (11) we then find, after a Wick rotation,

$$\int \mathscr{F} = \pi\eta\{D\}. \tag{12}$$

Even though we have derived (12) in the adiabatic approximation we know[17] that (modulo 2π, i.e., modulo spectral flow) it is exact for the operator (10). The η invariant is essentially the five-dimensional Chern-Simons secondary characteristic class ω_5,[17]

$$\eta\{D\} = i\int \omega_5(A) = (i/24\pi^3)\int dx\, \mathrm{Tr}\{A(dA)^2 + \tfrac{3}{5}A^5 + \tfrac{3}{2}A^3\, dA\}, \tag{13}$$

and in order to relate ω_5 to $\mathscr{F}$ we must first interpret it as a two-form on $\mathscr{A}^3$. For this we consider an arbitrary two-surface on $\mathscr{A}^3$ parametrized by u_1, u_2, i.e., with coordinates $A_i^a(x;u_1,u_2)$. On $\mathscr{A}^3$, (13) truncates to

$$\omega_5(A) = (1/24\pi^3)\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)A_i^a\,\partial_\rho A_j^b\,\partial_\sigma A_k^c\, dx^i dx^j dx^k du^\rho du^\sigma. \tag{14}$$

Upon integration of (14) over the physical three-space [recall that the original spatial coordinate $\mathbf{x}$ is an "index" for the "coordinate" $A_x = A_i^a(\mathbf{x})$ on $\mathscr{A}^3$] and since $\partial_\rho A\, du^\rho$ is the pull-back of the one-form $\mathbf{d}A$ on $\mathscr{A}^3$, from (12) and (13) we find, for the two-form $\mathscr{F}$,

$$\mathscr{F} = (i/24\pi^2)\epsilon^{ijk}\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)A_i^a\,\mathbf{d}A_j^b\,\mathbf{d}A_k^c. \tag{15}$$

Notice that since $\mathscr{F}$ is linear in the coordinate A_x the corresponding source is a constant [see (17) below]. Notice also that (15) vanishes unless the fermions are in the complex representation of the gauge group. Indeed, by analyzing the discrete symmetries of D we find that $\eta\{D\}$ can be nonzero only if the fermion representation is complex.

We shall now proceed to quantize the effective gauge-field Lagrangean: Consider the first sum in (6). In addition to $A_i^a(\mathbf{x},t)$ the Dirac Hamiltonian only involves *space* derivatives and consequently the eigenvalues $E_r(t)$ can only depend on t through the field $A_i^a(\mathbf{x},t)$ and its space derivatives. Since this sum does not contain time derivatives of $A_i^a(\mathbf{x},t)$, it can only contribute to the potential term of the effective Lagrangean. On the other hand, from (8) we find that the second sum in (6) is *linear* in the time derivatives of $A_i^a(\mathbf{x},t)$. Consequently, in the first-order formalism the effective Lagrangean obtained after evaluating the fermion determinant is of the form

$$L_{\text{eff}} = \left[-F_{it}^a - \langle \text{vac};\tau|\frac{1}{i}\frac{\delta}{\delta A_i^a}|\text{vac};\tau\rangle\right]\partial_t A_i^a + \mathscr{H}[A], \tag{16}$$

where $\mathscr{H}[A]$ is the Hamiltonian density whose details depend on the first term in (6). From (16) we find that the canonical momentum differs from the electric field,

$$\frac{\delta L_{\text{eff}}}{\delta\partial_t A_i^a} = \Pi_i^a = \frac{1}{i}\frac{\delta}{\delta A_i^a} = -E_i^a - \langle\text{vac};\tau|\frac{1}{i}\frac{\delta}{\delta A_i^a}|\text{vac};\tau\rangle.$$

Notice that this system is an infinite-dimensional version of a quantum mechanical point particle that moves in the background of a magnetic field[18] and in analogy with the finite-dimensional case covariant translations are generated by the "velocity" operator

$$E_x = i\frac{\delta}{\delta A_x} + i\langle\text{vac};\tau|\frac{\delta}{\delta A_x}|\text{vac};\tau\rangle = i\frac{\delta}{\delta A_x} + \mathscr{A}_x.$$

The commutator gives the "magnetic field," $[E_x,E_y] = i\mathscr{F}$, and the Jacobi identity gives the "magnetic source density,"

$$[E_x,[E_y,E_z]]+(\text{permutations})=-\mathbf{d}\mathscr{F}=-\frac{i}{12\pi^2}\epsilon^{ijk}\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)\mathbf{d}A_i^a\,\mathbf{d}A_j^b\,\mathbf{d}A_k^c. \tag{17}$$

Since the Jacobi identity fails, the covariant translations on $\mathscr{A}^3$ do not associate and cannot form a Lie group. However, since the right-hand side of (17) is constant the electric field forms a Malcev algebra,

$$[[E_x,E_y],[E_z,E_w]]=[E_x,[[E_w,E_y],E_z]]+[E_w,[[E_y,E_z],E_x]]+[E_y,[[E_z,E_x],E_w]]+[E_z,[[E_x,E_w],E_y]].$$

Elsewhere we shall show how this is generalized to an arbitrary number of dimensions.

In conclusion, we have presented a complete first-principles analysis of chiral anomalies. Since all phenomenologically interesting field-theory models contain Weyl fermions (for example, a Dirac fermion is a direct sum of two Weyl fermions), we expect that our approach will become useful in the analysis of realistic field-theory models. Finally, we have presented our results in the hope that a deeper understanding of the quantum structure of anomalous field theories will eventually lead to their consistent quantization.[19]

We thank S. Adler, O. Alvarez, L. Alvarez-Gaume, R. Dashen, B. Grossman, R. Jackiw, S. Jo, N. Seiberg, B. Simon, J. Stasheff, and M. Stern for helpful discussions. After this work was completed we received a Harvard University preprint by L. Alvarez-Gaume and S. Della-Pietra who have also found a relation between the effective action for the chiral gauge anomaly and η invariant. We thank L. Alvarez-Gaume for informing us about these results prior to publication. Subsequently, R. Jackiw has communicated to us a California Institute of Technology preprint by H. Sonoda who also discusses Berry's phase and the chiral gauge anomaly. This research was supported in part by the U.S. Department of Energy Grant No. DE-AC02-76ER02220 and by the Natural Sciences and Engineering Research Council of Canada.

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[1]See the articles in "Anomalies, Geometry and Topology," edited by A. White (World Scientific, Singapore, to be published).

[2]H. Georgi and S. Glashow, Phys. Rev. D **6**, 429 (1972); D. Gross and R. Jackiw, Phys. Rev. D **6**, 477 (1972).

[3]M. B. Green and J. H. Schwarz, Phys. Lett. **149B**, 117 (1984); D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. **54**, 502 (1985).

[4]B. Zumino, Y. S. Wu, and A. Zee, Nucl. Phys. **B239**, 477 (1984); B. Zumino, in *Relativity, Groups and Topology II,* Les Houches Summer School Proceedings, Session 40, edited by B. DeWitt and R. Stora (North-Holland, Amsterdam, 1984); A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).

[5]J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[6]W. A. Bardeen and B. Zumino, Nucl. Phys. **B244**, 421 (1984); B. Zumino, Nucl. Phys. **B253**, 477 (1985); L. Faddeev, Phys. Lett. **145B**, 81 (1984); L. Faddeev and S. Shatashvili, Teor. Mat. Fiz. **60**, 206 (1984) [Theor. Math. Phys. **60**, 770 (1984)]; J. Mickelsson, Commun. Math. Phys. **97**, 361 (1985).

[7]M. Atiyah and I. M. Singer, Proc. Natl. Acad. Sci. U.S.A. **81**, 2597 (1984); O. Alvarez, I. M. Singer, and B. Zumino, Commun. Math. Phys. **97**, 409 (1984); L. Alvarez-Gaume and P. Ginsparg, Nucl. Phys. **B243**, 449 (1984), and to be published.

[8]S. G. Jo, Massachusetts Institute of Technology Report No. MIT-CTP 1264, 1985 (unpublished).

[9]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D **17**, 2717 (1978); N. Christ, Phys. Rev. D **21**, 1591 (1980); A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **54**, 873 (1985).

[10]R. Jackiw, Massachusetts Institute of Technology Report No. MIT-CTP 1230, 1985 (unpublished); Y. S. Wu and A. Zee, to be published.

[11]A. Malcev, Mat. Sb. **78**, 569 (1955); A. A. Sagle, Trans. Amer. Math. Soc. **101**, 426 (1961); see also M. Günaydin and B. Zumino, Lawrence Berkeley Laboratory Report No. 19200, 1985 (unpublished).

[12]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **12**, 2443 (1975).

[13]M. V. Berry, Proc. Roy. Soc. London, Ser. A **392**, 45 (1984).

[14]B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).

[15]Faddeev, Ref. 6; Faddeev and Shatashvili, Ref. 6; P. Nelson and L. Alvarez-Gaume, Commun. Math. Phys. **99**, 103 (1985). (They also consider relations between anomalies and Berry's phase.)

[16]Notice that this implies that $\mathscr{A}_x$ cannot be a smooth functional. However, we find its *formal* use convenient as it provides a clear physical picture. Indeed, as Berry (Ref. 13) points out, $\gamma_r(\partial C)$ generally only makes sense as a surface integral and the sum of these surface integrals then defines $\mathscr{F}$.

[17]A. J. Niemi and G. W. Semenoff, Phys. Rev. D **30**, 809 (1984).

[18]For recent discussions, see Günaydin and Zumino, Ref. 11; R. Jackiw, Phys. Rev. Lett. **54**, 159 (1985), and to be published; B. Grossman, Phys. Lett. **152B**, 93 (1985), and to be published; Y. S. Wu and A. Zee, Phys. Lett. **152B**, 98 (1985).

[19]Faddeev and Shatashvili, Ref. 6.