Average Arrival Time of Wave Pulses through Continuous Random Media

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,[(a)] R. G. Frehlich, and Frank S. Henyey
Center for Studies of Nonlinear Dynamics,[(b)] *La Jolla Institute, La Jolla, California 92037*
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It is pointed out that a continuous random medium can cause an average advance of the arrival time of a pulse. This advance will occur for unsaturated and partially saturated propagation, but not in full saturation (which corresponds to the discrete-scatterer case). The effect, which is associated with Fermat's principle of least time, can be observed by measuring the difference between intensity-weighted and unweighted average arrival times.

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Wave packets, or pulses, are frequently used to probe inhomogeneous media. If the wave speed in the medium varies on scales small compared with the total distance traveled by the pulse, then the medium is treated by statistical methods. "Macroscopic" examples include sound through ocean internal waves,[1,2] light through atmospheric turbulence,[3] and radio waves through plasmas such as the ionosphere,[4] the solar wind,[5] or the interstellar medium.[6] Microscopic examples include various "sounds" through liquid helium,[7] and waves through inhomogeneous condensed matter.[8] These continuous media may be distinguished from media consisting of discrete scatterers such as would occur in light transmission through fog,[9] or in wave transmission through a gaslike medium with random-point particles.[10]

This Letter points out that a fluctuating continuous medium can cause an average advance of the pulse arrival time. All previous analyses have dealt with situations in which pulses are delayed on the average.[11,12] By convention,[1–12] the ensemble average of a random medium is taken as the medium reference state, and the small fluctuations about this reference state are thus by definition a zero-mean random process. The arrival-time advance or delay is relative to the travel time through the reference states. Thus, for example, results through turbulent air or plasma are relative to quiescent air or plasma, not vacuum.

The behavior of a wave propagating through a random medium is controlled by relationships between the wave number (k) of the propagating wave, the range (R), and the strength and size of the medium fluctuations.[1,2] *Unsaturated* behavior corresponds to one stationary-phase path (ray), and occurs if the medium fluctuations are weak enough. In *fully saturated* behavior the original ray breaks up into many new microrays which are statistically independent of each other. Propagation through a medium of discrete scatterers falls in this category. *Partially saturated* behavior occurs in a strongly fluctuating medium with a power-law spectrum, which has enough small-scale fluctuations to cause the breakup into many microrays, and enough large-scale fluctuations to make the microray bundle behave like a single ray in its wandering from the unperturbed ray. Experiments in waves propagating through continuous random media typically fall into this category. We deal only with the important case in which the transverse wandering from the unperturbed ray is small compared with the range of propagation.

Briefly our results are as follows: If the travel time of a pulse is averaged over an ensemble of the random medium, with each pulse weighted by its intensity, then the average pulse is delayed, regardless of the type of propagation behavior, in agreement with previous results.[11,12] However, if the average travel time is obtained without weighting by pulse intensity, then a pulse advance is expected for both unsaturated and partially saturated behavior, while a pulse delay remains for the fully saturated case. The difference between intensity-weighted and unweighted travel time probes the variance of the first derivative of the refractive index, smoothed over a microray bundle.

To explain our effect qualitatively we first take a simple special case. Consider a point source and point receiver separated by range R, and a homogeneous medium in the absence of fluctuations, so that the unperturbed ray from source to receiver is a straight line. The random medium is concentrated in a "phase screen" at a distance z from the source. This screen has the effect of advancing the time of a wave front by a random amount $t(x)$ where x is the position on the screen, and $t(x)$ is a stationary Gaussian random process with zero mean. (We take x as one-dimensional for simplicity.)

Weak fluctuations.—In the geometrical-optics limit only one ray exists from source to receiver. The travel time for a path through point x is

$$T(x) \approx T_0 + 0.5 c_0^{-1} A x^2 - t(x), \tag{1}$$

where $A^{-1} \equiv z(R-z)/R$. By Fermat's principle the ray is at the point x_r such that $T(x_r)$ is a minimum.

For the case of weak fluctuations we may expand $t(x)$ as

$$t(x)=t_0+t'x+0.5t''x^2. \quad (2)$$

The position of the ray follows to first order as

$$x_r=A^{-1}c_0t'. \quad (3)$$

The travel time of the ray is then

$$T(x_r)=T_0+0.5c_0A^{-1}t'^2-t_0-c_0A^{-1}t'^2. \quad (4)$$

This case requires that, typically

$$|c_0A^{-1}t'^2| << |t_0|. \quad (5)$$

But t (and hence t_0 and t') are (by construction) random variables with zero mean. Therefore the t_0 term will disappear in the *average* travel time and the only effect of the fluctuations will come from the t' terms. These terms arise because the ray has moved away from its unperturbed position. The first t' term is positive, corresponding to a pulse delay, and represents the effects of geometry; the perturbed path is physically longer than the unperturbed one. The second t' term is negative, corresponding to a pulse advance; we call this the Fermat term; the ray sought out a region of the medium with a pulse advance. The Fermat term is twice as large in magnitude as the geometry term. The average travel time is

$$\langle T\rangle=T_0-0.5c_0A^{-1}\langle t'^2\rangle, \quad (6)$$

so that the pulse on the average arrives early.

There is a subtlety to this result. In the weak-fluctuation limit the intensity is controlled by the focusing due to the curvature of the wave front as it exists from the phase screen. It is not difficult to show that the intensity I is, to first order,

$$I=1+A^{-1}c_0t''. \quad (7)$$

Consider the intensity-weighted average travel time

$$\langle IT(x_r)\rangle=T_0-0.5c_0A^{-1}\langle t'^2\rangle - c_0A^{-1}\langle t_0t''\rangle, \quad (8)$$

where the last term comes from the correlation between the intensity and the travel time. For any random function $t(x)$ whose Fourier components are uncorrelated (i.e., the correlation function is translation-invariant)

$$\langle t_0t''\rangle=-\langle t'^2\rangle. \quad (9)$$

Therefore

$$\langle IT(x_r)\rangle=T_0+0.5c_0A^{-1}\langle t'^2\rangle. \quad (10)$$

In other words, the intensity-weighted average travel time is *delayed* by fluctuations by exactly the amount that the unweighted average is advanced! The focusing effect exactly cancels the Fermat term, leaving a resultant equal to the geometry effect alone. This occurs because a positive fluctuation, which delays the pulse, acts as a converging lens to increase the intensity.

The simple example of a phase screen in the weak-fluctuation geometrical-optics limit has illustrated our point. We will now make some remarks on generalizations to extended media and strong fluctuations which we have treated rigorously but do not have space within the Letter format to describe in detail. We then describe a rigorous extension of these results by means of a path-integral method to include diffractive effects in a power-law medium.

There is no difficulty in extending the above results from a phase screen to extended media in which (6) and (10) are replaced by

$$\langle T\rangle-T_0=-0.5c_0^{-1}\int dz\,A^{-1}(z)\times\left[\int dz'\,\rho_{xx}(z,z')\right], \quad (11)$$

$$\langle IT\rangle-T_0=+0.5c_0^{-1}\int dz\,A^{-1}(z)\times\left[\int dz'\,\rho_{xx}(z,z')\right], \quad (12)$$

$$\rho_{xx}(z,z')=\langle\partial_x\mu(z)\partial x\mu(z')\rangle, \quad (13)$$

where $\partial_x\mu(z)$ is the transverse gradient of the refractive index due to the fluctuations at location z along the unperturbed ray. These results require the Markov approximation [that is, the quantity in square brackets in (11) is a local function of z]. If an incident plane wave rather than a point source is used, all three terms (geometry, Fermat, and focusing) are reduced by a factor of 3. If the Markov approximation is not made, the ratio between Fermat and geometry remains -2, while all terms are modified by terms of order L_p/R, where L_p is the longitudinal correlation length of the medium fluctuations.

Strong fluctuations.—If the medium fluctuations are strong enough, one can show in the limit of small wavelength that both the Fermat and focusing terms become negligible. This occurs when the intensity fluctuations become of order unity. From (7) this corresponds to

$$A^{-1}c_0\langle t''^2\rangle^{1/2}\gtrsim 1. \quad (14)$$

If this condition is satisfied, the unperturbed ray breaks up into many microrays that are not minima, but extrema, in accord with Hamilton's principle of stationary action. The converging (or diverging) lenses now are so strong that caustics occur, destroying the correlation between intensity and pulse delay. The Fermat term disappears because the microrays are not at global minima; hence the geometry term strongly dominates, resulting in the validity of (10) even for unweighted average travel time.

If $t(x)$ has a power-law spectrum, then the above argument must be modified to separate the effects of large-scale and small-scale fluctuations. The small-scale fluctuations, if they alone satisfy (14), break the unperturbed ray into a bundle of microrays of size L_μ. Fluctuations larger than L_μ correlate the travel times of all the microrays as though they were a single ray, and cause the average pulse arrival time to behave according to the weak-fluctuation rule (6), where $\langle t'^2\rangle$ is to be interpreted as coming only from scales larger than L_μ. It then becomes crucial to estimate L_μ. For the phase-screen case, we find

$$L_\mu \approx c_0 A^{-1}\langle t''^2\rangle^{1/2}. \tag{15}$$

The difficulty here is defining the average travel time of a pulse that itself has complicated structure due to microrays.

Path-integral result.—A major limitation of the above treatment is its restriction to cases of very small wavelength, and the related requirement of defining an inner scale for the medium fluctuations (in order for $\langle t''^2\rangle$ or even $\langle t'^2\rangle$ to be finite). The path-integral method can treat the case of finite wavelength for intensity-weighted travel time.

To arrive at average pulse travel time we note that

$$\langle IT\rangle - T_0 = -(i/c_0)\partial_\kappa\langle\psi^*(k_0+k)\psi(k_0)\rangle|_{k=0}, \tag{16}$$

where k_0 is a central wave number of the propagating wave, k is a deviation wave number whose excursion represents the pulse bandwidth, and ψ is the reduced wave function at the receiver. Let us treat the phase-screen case for simplicity. The second moment as a function of wave number is known to be[13,14]

$$\langle\psi^*(k_0+k)\psi(k_0)\rangle = \exp[-0.5k^2c_0^2\langle t_0^2\rangle]Q(k), \tag{17}$$

$$Q(k) = N\int du\,\exp\{\tfrac{1}{2}[iAk_0^2k^{-1}u^2 - k_0^2c_0^2D(u)]\}, \tag{18}$$

where $D(u)$ is the structure function of the medium,

$$D(u) \equiv \langle[t(u)-t(0)]^2\rangle, \tag{19}$$

and N is a normalization such that $Q(0)=1$. It is easy to see that $Q(k)$ has an increasing imaginary part as k grows from zero.

At very small k, $Q(k)$ is controlled by the behavior of $D(u)$ at small u, which must be quadratic if an inner scale exists. Thus for small u

$$D(u) = \langle t'^2\rangle u^2. \tag{20}$$

Then[14]

$$Q(k) = [1 - ic_0^2A^{-1}\langle t'^2\rangle k]^{-1/2}, \tag{21}$$

and the result for the intensity-weighted average travel time is (10) exactly.[15]

If we attempt to model $D(u)$ at infinitesimal u as a fractional power law, then the average travel time diverges.

This result for finite wave number predicts a delay equal to the geometry term. Most importantly this derivation has made no distinction between the weak- and strong-fluctuation cases, and can be easily generalized to the extended medium. We can conclude that $\langle IT\rangle - T_0$ is equal to the geometry term alone regardless of the fluctuation strength, a result already suggested by the geometrical-optics calculation.

When (18) is generalized to an extended medium by a path-integral formalism the same conclusions are easily shown, subject to the additional assumption of the Markov approximation.

An important modification of the above result occurs if, in the absence of fluctuations, the medium has focusing properties. In ocean sound propagation this is due to the sound channel. In radio-wave propagation from pulsars this might be due to very large-scale medium fluctuations that are effectively frozen during the time of observation. The modification can be simply expressed by generalizing $A^{-1}(z)$ in (11) and (12) for a curved ray.[2] The key result is that $A^{-1}(z)$ can be negative for various regions along the ray, and hence the *geometry term can be negative for curved rays.* This complication is crucial to the comparison between calculation and experiment in the ocean, though probably not in other media. Note that this effect provides another, different mechanism by which fluctuations in a medium may cause an average pulse advance. Finally, dispersive propagation, such as occurs for radio waves through plasma, can be treated with the same techniques.

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[a]On leave from the University of California, Santa Cruz, Cal. 95064.

[b]Affiliated with the University of California, San Diego, Cal. 92093.

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[15] Confusion has arisen becuase (20) for all u implies $t''=0$. The resolution is that limits taken with care imply that (9), and hence (10), remain true.