Quaternionic Quantum Field Theory

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We show that a quaternionic quantum field theory can be formulated when the numbers of bosonic and fermionic degrees of freedom are equal and the fermions, as well as the bosons, obey a second-order wave equation. The theory is initially defined in terms of a quaternion-imaginary Lagrangean using the Feynman sum over histories. A Schrödinger equation can be derived from the functional integral, which identifies the quaternion-imaginary quantum Hamiltonian. Conversely, the transformation theory based on this Hamiltonian can be used to rederive the functional-integral formulation.

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Ever since the classic work of Birkhoff and Von Neumann[1] on the foundations of quantum theory, there has been interest in the possibility of constructing a quaternionic generalization of complex quantum mechanics. Over the years a number of relevant mathematical and kinematical results have been obtained,[2] but the central problem of finding a viable dynamics for quaternionic quantum theory has remained unsolved. I now report progress on this problem.

My approach is based on the sum-over-histories method of Feynman,[3] and my basic kinematic tool is a remarkable formula which I recently discovered[4] for an oscillatory quaternionic Gaussian integral with equal numbers of bosonic and fermionic integration variables,

$$\lim_{\epsilon\to 0}\left(\prod_{i=1}^{M}\int d\phi^i d\psi^i\right)(4\pi^2)^{-M}\exp(-\bar{\phi}A\phi-\bar{\psi}B\psi+\bar{u}\phi-\bar{\phi}u+\bar{\xi}\psi+\bar{\psi}\xi+\mathscr{J}-\epsilon\bar{\phi}\phi)$$
$$=\det{}^2B\,\det{}^{-1}(A^\dagger A)\exp(-\bar{u}A^{-1}u+\bar{\xi}B^{-1}\xi+\mathscr{J}). \qquad (1)$$

The notation is as follows: $e_{1,2,3}$ are quaternionic imaginary units satisfying $e_ae_b=-\delta_{ab}+\epsilon_{abc}e_c$, with the conjugation operation $\bar{e}_a=-e_a$; ϕ and u are column vectors containing M real quaternions, and ψ and ξ are column vectors containing M real Grassmann quaternions (i.e., the ith components of ϕ and ψ are $\phi^i=\phi_0^i+\phi_a^ie_a$, $\psi^i-\psi_0^i+\psi_a^ie_a$, with $\phi_{0,a}^i$ real and $\psi_{0,a}^i$ real Grassmann), $B=B^\dagger\equiv\bar{B}^T$ and $A=-A^\dagger$ are $M\times M$ quaternionic matrices, $\mathscr{J}=-\bar{\mathscr{J}}$ is a fixed imaginary quaternion, the integration measure[5] is defined by $d\phi^i=d\phi_0^i\,d\phi_1^i\,d\phi_2^i\,d\phi_3^i$, $d\psi^i=d\psi_0^i\,d\psi_1^i\,d\psi_2^i\,d\psi_3^i$, and det denotes the Dyson-Moore[2] determinant. Apart from the infinitesimal convergence factor $\exp(-\epsilon\bar{\phi}\phi)$, the exponents on the left- and right-hand sides of Eq. (1) are quaternion imaginary. The exponential source dependence on the right is special to the case when the numbers of bosonic and fermionic integration variables are the same, and is an essential ingredient of the construction which follows.

The fundamental postulate, generalizing Dirac's observation[6] in the complex case, is that the quantum mechanical transformation function for an infinitesimal time interval $\Delta t=t_{j+1}-t_j$ has the form

$$\langle\{\phi_{j+1},\psi_{j+1}\},t_{j+1}|\{\phi_j,\psi_j\},t_j\rangle=C^{-1}\exp[\Delta t\tilde{L}(\{\phi_{j+1/2},\dot{\phi}_{j+1/2},\psi_{j+1/2},\dot{\psi}_{j+1/2}\},t_j+\Delta t/2)], \qquad (2)$$

with $\tilde{L}$ a quaternion-imaginary Lagrangean, $\tilde{L}=\tilde{L}_1e_1+\tilde{L}_2e_2+\tilde{L}_3e_3$ (analogous to iL in the complex case). The arguments of $\tilde{L}$ are to be evaluated in accordance with the trapezoidal rule at the midpoint of the interval. Compounding N infinitesimal transformations, we get for finite time evolution

$$\langle\{\phi_N,\psi_N\},t_N|\{\phi_0,\psi_0\},t_0\rangle$$
$$=\left[\prod_{j=1}^{N-1}\left(\prod_{i=1}^{M}\int d\phi_j^i\,d\psi_j^i\right)\right]C^{-1}\exp[\Delta t\tilde{L}(N-\tfrac{1}{2})]C^{-1}\exp[\Delta t\tilde{L}(N-\tfrac{3}{2})]\cdots C^{-1}\exp[\Delta t\tilde{L}(\tfrac{1}{2})], \qquad (3)$$

in an abbreviated notation where the exponent in Eq. (2) would be written as $\Delta t\tilde{L}(j+\frac{1}{2})$. In passing to the continuum limit, we must take into account the fact that the infinitesimal phases in Eq. (3) do not commute, and hence the product in Eq. (3) is not the exponential of the Riemann sum of exponents, as it is in the complex case. Formally, we can accomplish this by giving the quaternion units e_a a time label, $e_a\to e_a(t)$, and defining a time-ordering operation T as one which orders the e_a's with the later time on the left, giving the functional integral formula[7]

$$\langle \{\phi_N,\psi_N\},t_N|\{\phi_0,\psi_0\},t_0\rangle = \text{const}\times\int d[\phi]d[\psi]T\exp\left[\int_{t_0}^{t_N} dt\, e_a(t)\tilde{L}_a(t)\right]. \tag{4}$$

This construction guarantees that the transformation functions satisfy the quantum-mechanical composition law (the principle of superposition of wave amplitudes),

$$\langle \{\phi_N,\psi_N\},t_N|\{\phi_0,\psi_0\},t_0\rangle = \left[\prod_{i=1}^{M}\int d\phi_I^i\, d\psi_I^i\right]\langle \{\phi_N,\psi_N\},t_N|\{\phi_I,\psi_I\},t_I\rangle\langle \{\phi_I,\psi_I\}t_I|\{\phi_0,\psi_0\},t_0\rangle \tag{5}$$

for any arbitrary intermediate time t_I.

At this point let us assume a specific functional form for $\tilde{L}$,

$$\tilde{L}(\{\phi,\dot\phi,\psi,\dot\psi\},t) = \tilde{L}_{\text{kin}} - \tilde{V}(\{\phi,\psi\},t), \tag{6}$$

with

$$\tilde{L}_{\text{kin}} = \sum_{i=1}^{M}\left(\tfrac{1}{2}\bar{\dot\phi}^i e_3\dot\phi^i + \tfrac{1}{2}\bar{\dot\psi}^i\dot\psi^i\right), \tag{7}$$

as suggested by the choice $A=e_3$, $B=1$ in Eq. (1). The use of the quaternion unit e_3 in the first term of Eq. (7) is arbitrary; by the gauge transformation $\phi^i\rightarrow q\phi^i$ with q a constant quaternion with $\bar{q}q=1$, which is[4] an invariance of the integration measure $d\phi^i$, e_3 can be converted to the general quaternion imaginary unit $\bar{q}e_3q$. Hence, Eq. (7) is consistent with the principle of quaternion covariance enunciated by Finkelstein *et al.*,[2] which states that quaternionic quantum mechanics should not pick out a preferred quaternionic frame. (This would not be the case had we instead used $e_3\bar{\dot\phi}\dot\phi$ to get a quaternion-imaginary quantity, since the imaginary unit e_3 would then be unaffected by gauge transformations of ϕ.) The fermion kinetic term in Eq. (7) is unconventional in that it is second order in time derivatives (which need not[8] lead to difficulties.) The question of whether the results of this Letter can be extended to first-order fermion actions will be studied elsewhere.[9] Since a Grassmann quaternion satisfies $\psi^4=0$ (as contrasted with $\psi^2=0$ for a Grassmann complex number ψ), we expect quaternionic fermions to obey an unconventional statistics.

Let us now use the above formalism to derive the Schrödinger equation satisfied by the wave function

$$\Psi(\{\phi,\psi\},t)\equiv\langle\{\phi,\psi\},t|\Psi\rangle. \tag{8}$$

According to Eqs. (2)–(5), we have

$$\begin{aligned}\Psi(\{\phi,\psi\},t+\Delta t) &= \left[\prod_{i=1}^{M}\int d\phi_0^i\, d\psi_0^i\right]\langle\{\phi,\psi\},t+\Delta t|\{\phi_0,\psi_0\},t\rangle\langle\{\phi_0,\psi_0\},t|\Psi\rangle\\ &= \left[\prod_{i=1}^{M}\int d\phi_0^i\, d\psi_0^i\right]C^{-1}\exp[\Delta t\tilde{L}(\{\phi_{1/2},\dot\phi_{1/2},\psi_{1/2},\dot\psi_{1/2}\},t+\Delta t/2)]\Psi(\{\phi_0,\psi_0\},t),\end{aligned} \tag{9}$$

with

$$\phi_{1/2}^i=\tfrac{1}{2}(\phi^i+\phi_0^i),\quad \psi_{1/2}^i=\tfrac{1}{2}(\psi^i+\psi_0^i),\quad \dot\phi_{1/2}^i=(\phi^i-\phi_0^i)/\Delta t,\quad \dot\psi_{1/2}^i=(\psi^i-\psi_0^i)/\Delta t. \tag{10}$$

Making the change of variables $\phi_0^i=\phi^i+(2\Delta t)^{1/2}\eta^i$, $\psi_0^i=\psi^i+(2\Delta t)^{1/2}\zeta^i$, substituting Eqs. (6) and (7), performing a Taylor expansion of Ψ on left and right, and dropping terms of order $(\Delta t)^{3/2}$ and higher, we get (with indices a,b summed from 0 to 3)

$$\begin{aligned}&\Psi(\{\phi,\psi\},t)+\Delta t\,\partial\Psi(\{\phi,\psi\},t)/\partial t\\ &\quad=\left[\prod_{i=1}^{M}\int d\eta^i d\zeta^i\right]C^{-1}\exp\left[\sum_i(\bar\eta^i e_3\eta^i+\bar\zeta^i\zeta^i)-\Delta t\tilde{V}(\{\phi,\psi\},t)\right]\\ &\quad\times\left[1+\Delta t(\text{odd in }\eta,\zeta)+\Delta t\sum_{i,j}\left(\eta_a^i\frac{\partial}{\partial\phi_a^i}\eta_b^j\frac{\partial}{\partial\phi_b^j}+\zeta_a^i\frac{\partial}{\partial\psi_a^i}\zeta_b^j\frac{\partial}{\partial\psi_b^j}\right)\right]\Psi(\{\phi,\psi\},t).\end{aligned} \tag{11}$$

By comparison to Eq. (1), it is now clear that the normalization constant should be chosen[10] as $C=(4\pi^2)^M$, and the unit term in the Taylor expansion on the right gives

$$\exp[-\Delta t\tilde{V}(\{\phi,\psi\},t)]\Psi(\{\phi,\psi\},t)=[1-\Delta t\tilde{V}(\{\phi,\psi\},t)]\Psi(\{\phi,\psi\},t). \tag{12}$$

If we eliminate all terms odd in η_a^i and ζ_b^j, which integrate to zero, and use Eq. (1) to integrate out $M-1$ degrees of freedom, the remainder of the right-hand side of Eq. (11) takes the form

$$\Delta t\left\{I_{ab}^B\sum_i\frac{\partial}{\partial\phi_a^i}\frac{\partial}{\partial\phi_b^i}-I_{ab}^F\sum_i\frac{\partial}{\partial\psi_a^i}\frac{\partial}{\partial\psi_b^i}\right\}\Psi(\{\phi,\psi\},t),$$
$$I_{ab}^B=\int d\eta\, d\zeta(4\pi^2)^{-1}\eta_a\eta_b\exp(\bar{\eta}e_3\eta+\bar{\zeta}\zeta),\quad I_{ab}^F=\int d\eta\, d\zeta(4\pi^2)^{-1}\zeta_a\zeta_b\exp(\bar{\eta}e_3\eta+\bar{\zeta}\zeta). \tag{13}$$

The integrals in Eq. (13) can be evaluated by the techniques of Ref. 4, giving after some algebra the Schrödinger equation

$$\partial\Psi(\{\phi,\psi\},t)/\partial t=-\tilde{H}\Psi(\{\phi,\psi\},t), \tag{14}$$

with the $\tilde{H}$ the quaternion-imaginary Hamiltonian (analogous to iH in the complex case)

$$\tilde{H}=\tilde{H}_{\text{kin}}(\{\partial/\partial\phi,\partial/\partial\psi\})+\tilde{V}(\{\phi,\psi\},t),\quad \tilde{H}_{\text{kin}}\equiv-\frac{1}{6}\sum_{i=1}^{M}(\bar{D}_{\phi^i}e_3D_{\phi^i}+\bar{D}_{\psi^i}D_{\psi^i}),$$
$$D_\phi\equiv\partial/\partial\phi_0+e_1\,\partial/\partial\phi_1+e_2\,\partial/\partial\phi_2+e_3\,\partial/\partial\phi_3,\quad D_\psi\equiv\partial/\partial\psi_0+e_1\,\partial/\partial\psi_1+e_2\,\partial/\partial\psi_2+e_3\,\partial/\partial\psi_3. \tag{15}$$

As verification that Eqs. (1)–(15) define a consistent quantum-mechanical scheme, let us run the above derivation in reverse, and derive the functional integration formulation of Eqs. (1)–(7) from the transformation theory based on the Hamiltonian $\tilde{H}$. We first note that the argument leading to Eq. (12) tells us that for any smooth function f we have

$$\left\{\prod_{i=1}^{M}\int d\eta^i d\zeta^i\right\}(4\pi^2)^{-M}\exp\left[\sum_i(\bar{\eta}^ie_3\eta^i+\bar{\zeta}^i\zeta^i)-\Delta t\tilde{V}\right]f(\{(2\Delta t)^{1/2}\eta,(2\Delta t)^{1/2}\zeta\})$$
$$=\exp(-\Delta t\tilde{V})f(\{0,0\})+\Delta t\times(\tilde{V}\text{-independent terms})+O((\Delta t)^2)$$
$$=\left\{\prod_{i=1}^{M}\int d\eta^i d\zeta^i\right\}(4\pi^2)^{-M}\exp\left[\sum_i(\bar{\eta}^ie_3\eta^i+\bar{\zeta}^i\zeta^i)\right]\exp(-\Delta t\tilde{V})f(\{(2\Delta t)^{1/2}\eta,(2\Delta t)^{1/2}\zeta\}), \tag{16}$$

and so within a functional integral we have the equivalence

$$\exp(\Delta t\tilde{L}_{\text{kin}}-\Delta t\tilde{V})\leftrightarrow\exp(\Delta t\tilde{L}_{\text{kin}})\exp(-\Delta t\tilde{V}). \tag{17}$$

Applying the Trotter product formula[11] to the finite-time evolution operator $\exp(-\tilde{H}t)$ we get

$$\exp(-\tilde{H}t)=\{\exp[-(\tilde{H}_{\text{kin}}+\tilde{V})t/N]\}^N=[\exp(-\tilde{H}_{\text{kin}}t/N)\exp(-\tilde{V}t/N)+O(1/N^2)]^N, \tag{18}$$

and so taking $t/N=\Delta t$ we must prove that

$$\langle\{\phi,\psi\},t|\exp(-\Delta t\tilde{H}_{\text{kin}})\exp(-\Delta t\tilde{V})|\{\phi',\psi'\},t\rangle=(4\pi^2)^{-M}\exp(\Delta t\tilde{L}_{\text{kin}})\exp(-\Delta t\tilde{V}). \tag{19}$$

Without loss of generality[12] the coordinate eigenstates $|\{\phi',\psi'\},t\rangle$ can be taken to be quaternion real, $e_a|\{\phi',\psi'\},t\rangle=|\{\phi',\psi'\},t\rangle e_a$; they are then eigenstates of $\tilde{V}$ and $\exp(-\Delta t\tilde{V})$ can be moved outside the ket and factored away. We can now use translation invariance to set $\{\phi',\psi'\}=\{0,0\}$, and so we must prove

$$\langle\{\phi,\psi\},t|\exp(-\Delta t\tilde{H}_{\text{kin}})|\{0,0\},t\rangle=(4\pi^2)^{-M}\exp\left[\sum_{i=1}^{M}(\bar{\phi}^ie_3\phi^i+\bar{\psi}^i\psi^i)/2\Delta t\right]. \tag{20}$$

We do this by writing the left-hand side of Eq. (20) in the differential operator form

$$\exp[-\Delta t\tilde{H}_{\text{kin}}(\{\partial/\partial\phi,\partial/\partial\psi\})]\langle\{\phi,\psi\},t|\{0,0\},t\rangle. \tag{21}$$

The matrix element in Eq. (21) is just the δ function

$$\langle\{\phi,\psi\},t|\{0,0\},t\rangle=\delta(\phi,\psi)=\prod_{i=1}^{M}\delta(\phi_0^i)\delta(\phi_1^i)\delta(\phi_2^i)\delta(\phi_3^i)\psi_0^i\psi_1^i\psi_2^i\psi_3^i, \tag{22}$$

which by use of Eq. (1) with $A = ae_3$, $B = b$, $a, b \rightarrow 0$ can be shown to have the integral representation

$$\delta(\phi,\psi) = \left[\prod_{i=1}^{M} \int du^i \, d\xi^i\right] (2\pi)^{-4M} \cosh\Phi, \quad \Phi = \frac{1}{2} \sum_{i=1}^{M} (\bar{\phi}^i u^i - \bar{u}^i \phi^i + \bar{\psi}^i \xi^i + \bar{\xi}^i \psi^i). \tag{23}$$

Substituting the power-series expansion of Eq. (23) into Eq. (21) and commuting the differential operator $\tilde{H}_{\rm kin}$ through to the right gives

$$\langle \{\phi,\psi\}, t | \exp(-\Delta t \tilde{H}_{\rm kin}) | \{0,0\}, t \rangle$$
$$= \left[\prod_{i=1}^{M} \int du^i \, d\xi^i\right] (2\pi)^{-4M} \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left\{ \Phi^2 - \Delta t \left[\tilde{H}_{\rm kin}\left(\left\{\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \psi}\right\}\right), \Phi^2 \right] + O((\Delta t)^2) \right\}^{2n}. \tag{24}$$

A long and complicated algebraic calculation now shows that the integral in Eq. (24) reduces to

$$\left[\prod_{i=1}^{M} \int du^i \, d\xi^i\right] (2\pi)^{-4M} \exp\left[-\frac{1}{2} \Delta t \sum_{i=1}^{M} (\bar{u}^i e_3 u^i + \bar{\xi}^i \xi^i) + \Phi \right], \tag{25}$$

which by a final application of Eq. (1) yields the right-hand side of Eq. (20).

In conclusion, we emphasize that since the number of degrees of freedom M can be taken arbitrarily large, the formalism constructed above applies to the field-theory ($M \rightarrow \infty$) limit. Full details of the calculations outlined above, and a discussion of many related topics, will be published elsewhere.[9]

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