Symmetry Breaking in the Lattice Abelian Higgs Model

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A new gauge-invariant order parameter is introduced for the Abelian Higgs model and used to prove the existence of a phase transition for the lattice theory in three or more dimensions. In Landau gauge this order parameter is the limit of $\langle \phi(x)\bar{\phi}(y)\rangle$ as $|x-y| \to \infty$.

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In space-time dimension $d \geqslant 3$ the Euclidean lattice Abelian Higgs model is known to have two phases. In the QED phase, the photon is massless,[1] while in the Higgs phase the photon acquires a mass.[2,3] The standard explanation[4] of mass generation begins by the assumption that the U(1) symmetry is spontaneously broken and that the scalar field acquires a nonzero vacuum expectation value. This explanation suggests that the scalar field may have long-range order (LRO), as in a ferromagnet. In this paper we introduce a new gauge-invariant order parameter involving the scalar field. For a range of parameters inside the Higgs phase, we prove that this order parameter is nonzero. An easy argument shows that the order parameter is zero for a range of parameters inside the QED phase.

The scalar field is written as $\phi(x) = r(x)e^{i\theta(x)}$. For convenience we fix $r(x)$ to be a, so that

$$\tfrac{1}{2}|(D_\mu\phi)(x)|^2 = a^2\{1-\cos[\partial_\mu\theta(x) - eA_\mu(x)]\}.$$

[The lattice has unit spacing, and $(\partial_\mu f)(x) = f(x+e_\mu) - f(x)$, where e_μ is the μth unit vector.] The action is

$$S = \tfrac{1}{4}\sum_{\mu,\nu,x}|\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)|^2 + (2\alpha)^{-1}\sum_{\mu,x}|\partial_\mu A_\mu(x)|^2 - a^2\sum_{\mu,x}\cos[\partial_\mu\theta(x) - eA_\mu(x)],$$

where $A_\mu(x) \in (-\infty,\infty)$ and $\theta(x) \in [0,2\pi)$. We have included the standard gauge-fixing term; $\alpha = 1$ is Feynman gauge, and $\alpha = 0$ is Landau gauge.

To investigate whether $\phi(x)$ has LRO, a natural gauge-invariant observable to study is the lattice version of the string $\phi(x)\exp[ie\int_x^y \mathbf{A}\cdot d\mathbf{l}]\bar{\phi}(y)$. However, the expectation of this observable always decays exponentially in $|x-y|$ (see Fröhlich, Morchio, and Strocchi[5]). Order parameters based on the behavior of this observable have been proposed by Bricmont and Fröhlich,[6] and Fredenhagen and Marcu.[7]

Another way to make the two-point function of the scalar field gauge invariant is to introduce a smeared string as follows:

$$G(x,y) = \phi(x)\exp\left[-ie\sum_{\mu,z}A_\mu(z)h_\mu(z)\right]\bar{\phi}(y),$$

where h_μ is the electric field generated by charges $+1$ at x and -1 at y,

$$h_\mu(z) = \partial_\mu V(z-x) - \partial_\mu V(z-y).$$

The potential $V(z-w)$ is the kernel of $(-\Delta)^{-1}$, where Δ is the lattice Laplacian. Under a gauge transformation $\theta \to \theta + \chi$, $A_\mu \to A_\mu + e^{-1}\partial_\mu\chi$, the smeared string transforms as

$$e\sum A_\mu(z)h_\mu(z) \to e\sum A_\mu(z)h_\mu(z) + \chi(x) - \chi(y).$$

So $G(x,y)$ is gauge invariant. In Landau gauge $(\partial_\mu A_\mu = 0)$, the smeared string $\sum_{\mu,z}A_\mu(z)h_\mu(z)$ vanishes, so that $G(x,y)$ is just $\phi(x)\bar{\phi}(y)$.

Our main result concerns $\langle G(x,y)\rangle$, where the expectation is defined by the action S.

Theorem 1.—In $d \geqslant 3$, for a sufficiently large and e sufficiently small,

$$\langle G(x,y)\rangle \geqslant K > 0,$$

where K is independent of x and y.

By using a correlation inequality, we can show[8] that this result holds for a variable-length scalar field with the double-well potential $U(r) = \lambda(r^2-a^2)^2$, for any $\lambda > 0$.

When $e = 0$, Theorem 1 provides a new proof of LRO for the classical X-Y model in $d \geqslant 3$; for earlier proofs, see Frölich *et al.*[9,10]

By a correlation inequality,[11] $\langle G(x,y)\rangle$ for any choice of e is bounded above by the two-point function of the X-Y model at inverse temperature $\beta = a^2$. Thus $\langle G(x,y)\rangle$ decays exponentially if a is sufficiently small. This decay also holds if we use a single-well potential for the scalar field. Therefore, the model has a phase transition with order parameter $G_\infty = \lim_{|x-y|\to\infty}\langle G(x,y)\rangle$. This indicates that there is a region of parameters for which $\phi(x)$ has LRO in Landau gauge. We do not prove that this phase transition exactly coincides with the transition from a massless to a massive photon.

Our other main result is the following:

Theorem 2.—In $d \geqslant 3$ for ae sufficiently large and e

sufficiently small,

$$\langle G(x,y)\rangle = G_\infty + \frac{C}{|x-y|^{d-2}} + O\left(\frac{1}{|x-y|^{d-1}}\right),$$

where G_∞ is the order parameter defined above and C depends on a and e.

Theorem 2 says there are Goldstone bosons in our theory when it is interpreted as a model in statistical mechanics. However, this does not imply that there are massless particles in the physical Hilbert space of the corresponding quantum field theory.

The main ideas of our proof are sketched below. A detailed proof will appear elsewhere.[8] The proof begins by using the transformation of Balaban, Brydges, Imbrie, and Jaffe[3] to rewrite the noncompact model in a compact form. This gives $\langle G(x,y)\rangle = a^2 Z(h)/Z(0)$, where

$$Z(h) = \sum_v \int DA \exp\left\{-\tfrac{1}{4}\sum_{\mu,\nu,z} |\partial_\mu A_\nu(z) - \partial_\nu A_\mu(z) + \frac{2\pi}{e} v_{\mu\nu}(z)|^2 + a^2 \sum_{\mu,z}\cos[eA_\mu(z)]\right\} \times \exp\left[-ie\sum_{\mu,z} A_\mu(z) h_\mu(z) - 2\pi i \sum_{\mu,z} n_\mu^v(z) h_\mu(z)\right].$$

Each $A_\mu(z)$ is integrated from 0 to $2\pi/e$. The sum is over closed, integer-valued two-forms v (in the usual notation, v is closed if $dv=0$). n^v is an integer-valued one-form chosen so that $\partial_\mu n_\nu^v - \partial_\nu n_\mu^v = v_{\mu\nu}$ (or $dn^v = v$).

If e is small, the term $a^2\cos[eA_\mu(z)]$ produces a mass ae for $A_\mu(z)$, and nonzero v are strongly suppressed. So to leading order in e,

$$Z(h) \approx \int DA \exp\left[-\tfrac{1}{4}\sum_{\mu,\nu,z}|\partial_\mu A_\nu(z) - \partial_\nu A_\mu(z)|^2 - \tfrac{1}{2}a^2e^2\sum_{\mu,z} A_\mu^2(z)\right] \exp\left[-ie\sum_{\mu,z} A_\mu(z) h_\mu(z)\right].$$

If a is large, the restriction on the range of integration of A can be neglected. The integral is now Gaussian and can be computed. This gives

$$\frac{Z(h)}{Z(0)} \approx \exp\left[-\tfrac{1}{2}e^2 \sum_{\mu,\mu',z,z'} h_\mu(z) C_{\mu\mu'}(z-z') h_{\mu'}(z')\right] \geqslant \exp\left[-(2a^2)^{-1}\sum_{\mu,z}|h_\mu(z)|^2\right], \tag{1}$$

since as an operator $C \leqslant (ae)^{-2}$. The electric field $|\partial_\mu V(r)|$ is asymptotically r^{-d+1}, so that $\sum_{\mu,z}|h_\mu(z)|^2$ is bounded uniformly in x and y when $d \geqslant 3$.

To make this bound rigorous, we perform a cluster expansion for $Z(h)$. In Ref. 3 an expansion was developed about the Gaussian measure with covariance C. Our expansion is about the product measure $\exp\{a^2\sum_{\mu,z}\cos[eA_\mu(z)]\}DA$. For this technically simpler expansion to converge, we need $(ae)^2$ to be large, and e small. Correlation inequalities[11] extend the result to the range of parameters in Theorem 1. Presumably, the expansion of Ref. 3 could be adapted to enlarge the range of parameters for which Theorem 1 holds.

The massless decay of the truncated correlation (Theorem 2) can be easily seen from the approximation (1). In this approximation

$$G_\infty \approx a^2 \exp\left[-e^2 \sum_{\mu,\mu',z,z'} \partial_\mu V(z) C_{\mu\mu'}(z-z') \partial_{\mu'} V(z')\right],$$

since V is translation invariant. Therefore,

$$\langle G(x,y)\rangle - G_\infty \approx G_\infty\left\{\exp\left[e^2 \sum_{\mu,\mu',z,z'} \partial_\mu V(z-x) C_{\mu\mu'}(z-z') \partial_{\mu'} V(z'-y)\right] - 1\right\}.$$

In $d \geqslant 3$,

$$\sum_{\mu,\mu',z,z'} \partial_\mu V(z-x) C_{\mu\mu'}(z-z') \partial_{\mu'} V(z'-y) \approx \frac{1}{a^2e^2}\frac{1}{|x-y|^{d-2}},$$

and so

$$\langle G(x,y)\rangle - G_\infty \approx G_\infty \frac{1}{a^2}\frac{1}{|x-y|^{d-2}}.$$

The rigorous proof of this decay uses the cluster expansion, which also gives higher-order corrections.

To summarize, we have found a gauge-invariant order parameter for the Higgs transition which indicates that the standard picture of spontaneous symmetry breakdown is valid in Landau gauge. This property is not shared by any other α gauge if $d \leq 4$; when $\alpha \neq 0$ the two-point function $\langle \phi(x) \overline{\phi}(y) \rangle$ decays to zero for $d \leq 4$ (Ref. 8).

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