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Structure of Constrained Hamiltonian Systems and Becchi-Rouet-Stora Symmetry

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The structure of constrained Hamiltonian systems is investigated. The structure functions are shown to obey remarkable identities which ensure the existence of the Becchi-Rouet-Stora symmetry for any gauge theory.

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Constrained Hamiltonian systems, whose systematic study has been initiated by Dirac,[1] provide an important tool in the investigation of gauge theories. Not only do they constitute an essential ingredient in the justification of the Faddeev-Popov representation of the path integral,[2] but, as it has been realized recently, they also play a key role in the understanding of anomalies.[3]

It is the purpose of this Letter to point out that constrained Hamiltonian systems possess a rich structure, characterized by "structure functions" obeying a series of remarkable identities. This classical structure has gone unnoticed so far because it is trivial for gauge theories with a closed algebra. However, in the "open" algebra case—to be defined more precisely below—this structure displays all its facets. Because of its relevance to the quantum theory, it seems accordingly indispensable to get familiar with it.

At the bottom of the "ladder" of structure functions, one finds the "zeroth-order structure functions." These are just the constraints themselves, which will be denoted by $\phi_\alpha(q,p) = U_\alpha^{(0)}(q,p)$. Here (q^i,p_i) are the canonical variables, which may include the Lagrange multipliers and their (weakly zero) conjugate momenta, and which are taken to be commuting for notational simplicity.

The meaning of the zeroth-order structure functions is well known: On the one hand, they restrict the classically available portion of phase space by the equations $\phi_\alpha(q,p) \approx 0$. On the other hand, they are the canonical generators of the gauge transformations. For this latter reason, their actual form can be usually guessed directly from the given gauge transformations without an explicit knowledge of the Lagrangean. At the quantum level, the constraint equations $\hat{\phi}_\alpha|\psi\rangle = 0$ express the gauge invariance of the theory in the physical subspace.[1,4]

Because the degeneracy of the Lagrangean is assumed to be solely due to the gauge invariance, all the constraints are first class,[1]

$$[\phi_\alpha, \phi_\beta] = -2U_{\alpha\beta}^{(1)\rho}\phi_\rho. \tag{1}$$

The constraint surface is accordingly invariant under the gauge transformations. In (1), the functions $U_{\alpha\beta}^{(1)\rho} = -U_{\beta\alpha}^{(1)\rho}$ are the "first-order structure functions" (the factor -2 has been inserted for further convenience). These functions may depend on the canonical variables, in which case one says that the gauge "algebra" is "open." The commutator of two gauge transformations generated by the constraints is then a transformation of the same type only on the constraint surface (this commutator involves indeed the term $[\ldots, U_{\alpha\beta}^{(1)\gamma}]\phi_\gamma$, which only vanishes weakly).

In the Yang-Mills case, the first-order structure functions are equal to the gauge-group structure constants (up to the numerical factor $-\frac{1}{2}$). In gravity, they reveal the structure of the surface deformation "algebra."[5] More generally, they are related in a simple way to the variation of the Lagrange multipliers under a gauge transformation, which relationship provides a means to compute them without the explicit form of the constraints.[6]

From the Jacobi identity for the Poisson brackets, it follows that $\sum[[\phi_\alpha, \phi_\beta], \phi_\gamma]=0$, where one sums over all cyclic permutations of (α, β, γ). This implies, when (1) is taken into account, that

$$\phi_\delta\{\sum([U_{\alpha\beta}^{(1)\delta}, \phi_\gamma]+2U_{\alpha\beta}^{(1)\lambda}U_{\gamma\lambda}^{(1)\delta})\}=0. \tag{2}$$

Now, contrary to what is sometimes (wrongly) asserted, it does not result from this identity that the coefficient of ϕ_δ in (2) vanishes. Rather, one gets

$$[U_{(\alpha\beta}^{(1)\delta}, \phi_{\gamma)}]+2U_{(\alpha\beta}^{(1)\lambda}U_{\gamma)\lambda}^{(1)\delta}=2U_{\alpha\beta\gamma}^{(2)\rho\sigma}\phi_\sigma, \tag{3}$$

where the functions $U_{\alpha\beta\gamma}^{(2)\rho\sigma}(q,p)$ are antisymmetric in (ρ, σ) and (α, β, γ), and will be named "second-order structure functions" [the subscript parentheses in (3) denote, as usual, complete antisymmetrization]. Although these functions are easily seen to vanish for Yang-Mills theories (as a result of the Jacobi identity for the structure constants) and gravity, they are nonzero in the generic case (e.g., $N=1$ supergravity[7] and relativistic membrane,[8] both without auxiliary fields) and, hence, they cannot be neglected.

Since equations of the same type as (2),

$$\lambda^\alpha\phi_\alpha=0, \tag{4}$$

will repeatedly appear in the sequel, it is worth discussing them briefly. These equations are local, algebraic, and linear in both λ^α and ϕ_α. Off the constraint surface, one easily sees by the standard methods of linear algebra that their general solution is

$$\lambda^\alpha=\mu^{\alpha\beta}\phi_\beta, \tag{5}$$

where $\mu^{\alpha\beta}$ is antisymmetric in (α, β) but is otherwise arbitrary [make a linear transformation to a new "frame" in which $\phi_\alpha=(1,0,0,\ldots,0)$]. When λ^α is sufficiently well behaved near $\phi_\alpha\approx 0$, and provided the constraints are independent ("irreducible") —which I assume—the solution (5) remains valid in the vicinity of and on the constraint surface. Conversely, if λ^α is given, and if one regards the equations (5) as equations for the unknown $\mu^{\alpha\beta}$, a necessary and sufficient condition for the existence of $\mu^{\alpha\beta}$ is that (4) holds. In that case, $\mu^{\alpha\beta}$ is determined up to the addition of $\nu^{\alpha\beta\gamma}\phi_\gamma$, where $\nu^{\alpha\beta\gamma}$ is an arbitrary, completely antisymmetric "tensor."

It turns out that the second-order structure functions $U_{\alpha\beta\gamma}^{(2)\rho\sigma}$ also obey an identity of the form (2), which enables one, as the above discussion of (4) shows, to define the structure functions of order three. Along exactly the same lines, one can then climb the "ladder" of the structure functions and define successively the fourth-, fifth-, . . . order structure functions.

More explicitly, one finds the following identity for the structure functions of order $\leq n$:

$$D^{(n)}(U)^{(\alpha_1\cdots\alpha_n)}_{(\beta_1\cdots\beta_{n+2})}\phi_{\alpha_n}=0, \tag{6}$$

where $D(U)$ is given by

$$D^{(n)}(U)^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_{n+2}}=\frac{1}{2}\sum_{q=0}^{n}[U^{(q)\alpha_1\cdots\alpha_q}_{\beta_1\cdots\beta_{q+1}}, U^{(n-q)\alpha_{q+1}\cdots\alpha_n}_{\beta_{q+2}\cdots\beta_{n+2}}](-)^{nq+1}$$
$$-\sum_{q=0}^{n-1}(q+1)(n-q+1)U^{(q+1)\alpha_1\cdots\alpha_q\gamma}_{\beta_1\cdots\beta_{q+2}}U^{(n-q)\alpha_{q+1}\cdots\alpha_n}_{\beta_{q+3}\cdots\beta_{n+2}\gamma}(-)^{n(q+1)}. \tag{7}$$

This identity implies, from our analysis of the equation (4), that the coefficients $D^{(n)}(U)$ in (6) are combinations of the constraints, i.e., that one has

$$D^{(n)}(U)^{(\alpha_1\cdots\alpha_n)}_{(\beta_1\cdots\beta_{n+2})}=(n+1)U^{(n+1)\alpha_1\cdots\alpha_n\alpha_{n+1}}_{\beta_1\cdots\beta_{n+2}}\phi_{\alpha_{n+1}}. \tag{8}$$

Equation (8) defines the structure functions $U^{(n+1)}$ of order $(n+1)$ which are completely antisymmetric in $(\beta_1, \ldots, \beta_{n+2})$ and $(\alpha_1, \ldots, \alpha_{n+1})$.

It is important to emphasize here that the defining equation (8) of $U^{(n+1)}$ only makes sense because of the remarkable identity (6). Without it, there would be no solution to (8) and there would be no "structure functions" of order $n+1$. Hence, the entire ladder of the structure functions leans on the identity (6), which can be considered as the main result reported in this Letter.

The identity (6)—which we know is true for $n=0,1$—is derived by induction as follows: (i) one takes the Poisson bracket of the defining equation of the $(n+1)$th order structure functions with the constraints ϕ_α and one completely antisymmetrizes the resulting expression with respect to the lower indices; (ii) one systematically uses the Jacobi identity for the Poisson brackets and the defining equations (8) of order $\leq n$ to transform the expression obtained in (i) into the desired form (6), with n replaced by $n+1$.

The calculations are relatively involved but present only technical difficulties. They will be given in all their glorious details in a forthcoming paper by the author,[9] which also treats the case of anticommuting constraints, discusses the ambiguity in $U^{(n+1)}$, and points out that the identity (6) reflects the possibility to transform (at least locally) any set of first-class constraints into equivalent Abelian ones.

The existence of (in general) nontrivial structure functions is a purely classical result which has apparently gone unnoticed so far and which has in itself some intrinsic interest. Presumably, it deserves further study. However, the relevance of the structure functions to the physical world goes much beyond the classical domain: The functions play a key role in the quantum theory through the Becchi-Rouet-Stora (BRS) symmetry,[10] as I now proceed to discuss.

It has emerged recently that quantum gauge fields possess a remarkable invariance of the supersymmetry type, called BRS symmetry. This invariance leads to the Ward identity. Its generator Ω is an essential building block of the quantum effective action.[11,12] Furthermore, in the "big" Hilbert space containing transverse, longitudinal, and ghost states, the physical subspace is simply defined as that subspace with zero BRS charge,[13]

$$\hat{\Omega}\,|\psi\rangle = 0. \tag{9}$$

The anomalies possess also a transparent BRS interpretation.[10,14,15] For these reasons, one can argue that BRS symmetry is the fundamental symmetry of quantum gauge fields, from which everything else follows.[15]

A key property of the BRS generator is its nilpotence, $\hat{\Omega}^2 = 0$. If one deals with the classical analog Ω of $\hat{\Omega}$, this condition reads[11]

$$[\Omega, \Omega] = 0, \tag{10}$$

since, for anticommuting generators, the Poisson bracket is symmetric and corresponds to the quantum anticommutator. Without the nilpotence of Ω, the quantum theory would be inconsistent.

It is convenient for what follows to expand Ω in powers of the ghost canonical variables η^α, $\mathcal{P}_\alpha$ (one such canonical pair per constraint),

$$\Omega = \sum_{n \geqslant 0} \eta^{\alpha_{n+1}} \cdots \eta^{\alpha_1} \Omega^{(n)\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_{n+1}} \mathcal{P}_{\beta_n} \cdots \mathcal{P}_{\beta_1}. \tag{11}$$

The coefficients $\Omega^{(n)}$ do not depend on the ghosts. Moreover, there is always one more η^α than there are $\mathcal{P}_\alpha$ in each term of (11), so that Ω is an odd element of the Grassmann algebra and has ghost number $+1$.

In order to make contact with gauge invariance, one also demands that

$$\Omega^{(0)}_\alpha = \phi_\alpha. \tag{12}$$

This ensures that to lowest order in the ghosts, Ω generates gauge transformations in which the infinitesimal parameters ϵ^α are replaced by the ghost fields η^α.

So the question is this: Is it possible to fulfill the nilpotency condition (10) and the "initial" conditions (12) with the expansion (11)? This question has been addressed long ago by Fradkin and his collaborators in pioneer papers,[11,16] in which they show that (10) implies the following equations on $\Omega^{(n+1)}$ $(n \geqslant 0)$:

$$D^{(n)}(\Omega)^{(\alpha_1 \cdots \alpha_n)}_{(\beta_1 \cdots \beta_{n+2})} = (n+1)\,\Omega^{(n+1)\alpha_1 \cdots \alpha_n \alpha_{n+1}}_{\beta_1 \cdots \beta_{n+2}} \phi_{\alpha_{n+1}}. \tag{13}$$

The quantity $D^{(n)}(\Omega)$ appearing in (13) has the same form as $D^{(n)}(U)$ [Eq. (7)], with $U^{(q)}$ replaced everywhere by $\Omega^{(q)}$. Thus, the conclusion of Ref. 16 is that Ω exists provided one can find a solution to Eqs. (13). However, to the author's knowledge, neither the existence of solutions to (13) nor the uniqueness problem has been discussed, so that the construction of the BRS generator in the general case of arbitrary first-class constraints stands so far on an incomplete, heuristic basis (as also pointed out by Batalin and Vilkovisky[17]).

It is here that the structure functions come into play. If one compares (13) with the defining equations (8) and notices that the condition (12) means that the $\Omega^{(0)}_\alpha$ are just the zeroth-order structure functions, one concludes that

$$\Omega^{(n)\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_{n+1}} = U^{(n)\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_{n+1}} \tag{14}$$

(to all orders). The structure functions are accordingly the coefficients of the BRS generator in an expansion in powers of the ghosts. The existence of these structure functions, which relies on the identity (6) demonstrated here, yields therefore an existence proof of Ω and makes it possible (in principle) to quantize any gauge theory along BRS lines.[18] Equations (11) and (14)—and the remarks below Eq. (2)—also clearly indicate that the BRS generator Ω contains, in general, multighost interactions of order greater than 3.

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Note added.—After completion of this work, I was kindly informed by I. A. Batalin that the "Moscow group" had also obtained, some time ago, the existence proof of the BRS transformation in the general case. In that respect, one cannot stress enough the importance of the remarkable work by Fradkin and his collaborators, which possesses an enormous range of application.

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