Generalized Supersymmetric Quantum Mechanics

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The spectrum of a supersymmetric quantum mechanical theory that leads to a generalized superalgebra is computed exactly. It has the feature that both the ground and first excited levels have unequal numbers of Bose and Fermi states.

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Supersymmetry[1] is increasingly emerging as the main new concept that theoretical physicists are using in their efforts to understand nature. A lot of attention has been focused on the techniques of breaking supersymmetry in ways that do not compromise the good quantum properties of supersymmetric theories.[2,3] In this Letter, I describe a supersymmetric quantum mechanical system that possesses the novel and paradoxical feature that, while the Hamiltonian is supersymmetric, both the ground *and* the first excited energy levels have unequal numbers of bosonic and fermionic states. The resolution of this paradox is contained in the supersymmetry algebra which turns out to be somewhat different from the algebra usually encountered in supersymmetric quantum mechanics.[2,4] If this feature generalizes to field theories, it could provide a way to account for the observed asymmetry in the low-energy spectrum of bosons and fermions in the real world that does not necessitate a supersymmetry breaking in the way usually envisaged.

The quantum mechanical system that we shall study is the supersymmetric CP^1 model[5] modified by the addition of a supersymmetric Wess-Zumino (WZ) term.[6] For orientational purposes I point out that in the O(3) sigma model, and therefore in the equivalent CP^1 model, a WZ term corresponds to a background Dirac monopole field.[7] The $N=\frac{1}{2}$ supersymmetric version[8] of this model corresponds to a spin-$\frac{1}{2}$ particle with its spin constrained to be tangential to the two-sphere around the monopole.[9] We shall, however, principally be interested in the $N=1$ supersymmetric version here, although in the process we shall also solve the $N=\frac{1}{2}$ version.

The CP^1 model has two complex fields Z^α, $\alpha=1,2$, and their superpartners, the two two-component complex spinors $\psi^{\alpha a}$. $a=1,2$ is the "Dirac" index. The fields satisfy the constraints $\bar{Z}Z=1$ and $\bar{Z}\psi=Z\bar{\psi}=0$. The Lagrangean of the CP^1 model is obtained by dimensional reduction from the two-dimensional field theory given in Ref. 5,

$$L=\overline{D_tZ}D_tZ+i\bar{\psi}\boldsymbol{D}\psi+\tfrac{1}{4}[(\bar{\psi}\psi)^2+(\bar{\psi}\gamma_5\psi)^2-(\bar{\psi}\gamma_0\psi)^2-(\bar{\psi}\gamma_1\psi)^2], \tag{1}$$

where $D_t=\partial_t-\bar{Z}\partial_tZ$ and $\gamma^0=\sigma_D^3$, $\gamma^1=\sigma_D^1$, and $\gamma^5=i\sigma_D^2$. (The subscript D indicates that the matrices act on the Dirac index a). The supersymmetry transformations are

$$\delta Z^\alpha=i\bar{\epsilon}\psi^\alpha,\quad \delta\psi^\alpha=-\tfrac{1}{2}i\epsilon Z^\alpha(\bar{\psi}\psi)+\tfrac{1}{2}i\gamma_5\epsilon Z^\alpha(\bar{\psi}\gamma_5\psi)+\gamma_5\epsilon[D_tZ^\alpha-\tfrac{1}{2}iZ^\alpha\bar{\psi}\gamma^0\psi]-\tfrac{1}{2}i\gamma_1\epsilon[Z^\alpha\bar{\psi}\gamma_1\psi]. \tag{2}$$

Following Ref. 7 the WZ term for the CP^1 model is $i\int dt\,\bar{Z}\,\overleftrightarrow{\boldsymbol{\partial}}_tZ$. The supersymmetric version is then easily shown to be $\int dt(i\bar{Z}\,\overleftrightarrow{\boldsymbol{\partial}}_tZ+2\psi^\dagger\psi)$ and its coefficient, m, is quantized—this is the usual Dirac quantization of the monopole charge. The Lagrangean (1) with the WZ term can be rewritten with an auxiliary field A as

$$L=\overline{D_t'Z}D_t'Z+i\bar{\psi}\boldsymbol{D}'\psi+\tfrac{1}{4}[(\bar{\psi}\psi)^2+(\bar{\psi}\gamma_5\psi)^2-(\bar{\psi}\gamma_1\psi)^2]-2mA+m\psi^\dagger\psi+m^2, \tag{3}$$

where $D_t'=\partial_t-iA$ and A is determined by its equation of motion to be $-\frac{1}{2}(i\bar{Z}\,\overleftrightarrow{\boldsymbol{\partial}}_tZ+\psi^\dagger\psi-2m)$. To quantize the system we proceed to work in the Hamiltonian formalism after going to the "temporal $A=0$ gauge." Thus we have the classical Hamiltonian

$$H=\partial_t\bar{Z}\partial_tZ-\tfrac{1}{4}[(\bar{\psi}\psi)^2+(\bar{\psi}\gamma_5\psi)^2-(\bar{\psi}\gamma_1\psi)^2]-m\psi^\dagger\psi-m^2, \tag{4}$$

with Gauss's law as an additional constraint, $i\bar{Z}\,\overleftrightarrow{\boldsymbol{\partial}}_tZ+\psi^\dagger\psi=2m$.

Now, using Dirac's procedure for constrained Hamiltonian systems,[10] we can set up the commutation and anticommutation relations. There are some ordering ambiguities that arise in doing this. They can be resolved by going to the $m=0$ case and then requiring that the usual supersymmetry algebra be satisfied. The details of the calculation will be presented elsewhere.[11] The result for the nonvanishing commutators and anticommutators is

$$[Z^\alpha,P_\beta]=i(\delta^\alpha_\beta-Z^\alpha\bar{Z}_\beta/2),\quad [\bar{Z}_\beta,\bar{P}^\alpha]=i(\delta^\alpha_\beta-Z^\alpha\bar{Z}_\beta/2),\quad [Z^\alpha,\bar{P}^\beta]=-iZ^\alpha Z^\beta/2,\quad [\bar{Z}_\alpha,P_\beta]=-i(\bar{Z}^\alpha\bar{Z}_\beta/2),$$
$$[P_\alpha,P_\beta]=-i\bar{Z}_\alpha P_\beta/2+i\bar{Z}_\beta P_\alpha/2,\quad [\bar{P}^\alpha,\bar{P}^\beta]=-iZ^\alpha\bar{P}^\beta/2+iZ^\beta\bar{P}^\alpha/2,\quad [P_\alpha,\bar{P}^\beta]=\tfrac{1}{2}i(Z^\beta P_\alpha-\bar{Z}_\alpha\bar{P}^\beta)-\psi^\dagger_\alpha\psi^\beta,$$
$$[P_\alpha,\psi^\dagger_\beta]=i\psi^\dagger_\alpha\bar{Z}_\beta,\quad [\bar{P}^\alpha,\psi^\beta]=i\psi^\alpha Z^\beta,\quad \{\psi^{\alpha a},\psi^{\dagger b}_\beta\}=\delta^{ab}(\delta^\alpha_\beta-Z^\alpha\bar{Z}_P), \tag{5}$$

where P_α is the momentum conjugate to Z^α. The supersymmetry charge Q^a is given by $\psi^{\alpha a}P_\alpha$, and the supersymmetry algebra for the $m=0$ case is

$$\{Q^a,Q^{\dagger b}\}=\delta^{ab}H,\quad [Q^a,H]=0, \tag{6}$$

where $H=(\bar{P}P+P\bar{P})/2-\frac{1}{4}(\psi^\dagger\sigma^i_D\psi)^2-\frac{3}{2}\psi^\dagger\psi$.[12] We can also define an angular momentum generator J^i satisfying $[J^i,J^j]=i\epsilon^{ijk}J^k$:

$$J^i=\tfrac{1}{2}i(Z\sigma^iP-\bar{Z}\sigma^i\bar{P}+i\psi^\dagger\sigma^i\psi). \tag{7}$$

We then recover the results of Davis, Macfarlane, and van Holten[13] for the supersymmetric O(3) nonlinear sigma model, namely, $H=J^2$ and $Z\sigma^i\bar{Z}J^i=\psi\psi^\dagger-1$.[14] Since $[\psi\psi^\dagger,H]=0$ the eigenvalues n of $\psi\psi^\dagger$ are good quantum numbers and can be used to label the states. n can be 0, 1, or 2. To see this, go to a coordinate patch where $Z_2\neq 0$. We can then express ψ^a_1 in terms of ψ^a_2 by solving $\bar{Z}\psi=0$. Thus we are left with ψ^1_2 and ψ^2_2 as the two independent fermionic operators. If we let $|0,0\rangle$ be the bosonic "vacuum" state satisfying $\psi^{a\dagger}_2|0,0\rangle=0$ (i.e., $n=0$), then $\psi^1_2|0,0\rangle=|1,0\rangle$ and $\psi^2_2|0,0\rangle=|0,1\rangle$ are two fermionic states that have $n=1$, and $\psi^1_2\psi^2_2|0,0\rangle=|1,1\rangle$ is a bosonic state having $n=2$. The energy levels are given by $j(j+1)$ with $j\geq|n-1|$. The spectrum is as shown in Fig. 1. There are two fermionic states of zero energy and the rest of the energy levels have equal numbers of Bose and Fermi states. Supersymmetry is unbroken because the ground state is annihilated by both Q^1 and Q^2.

We now turn to the case where $m\neq 0$. First consider the simpler model with $N=\frac{1}{2}$ supersymmetry. This is obtained by a consistent truncation of the $N=1$ model: $\psi^2_\alpha=0$ and in the supersymmetry transformations the parameter ϵ^2 is set equal to zero. We find in that case

$$H=\{Q,Q^\dagger\}=(P\bar{P}+\bar{P}P)/2-\tfrac{1}{4}(\psi^\dagger\psi)^2-m\psi^\dagger\psi-m(m+1)=J^2-m(m+1), \tag{8}$$

where $J^i=\frac{1}{2}(Z\sigma^iP-\bar{Z}\sigma^i\bar{P}+\frac{1}{2}\psi^\dagger\sigma^i\psi)$ and $Z\sigma^i\bar{Z}J^i=\psi\psi^\dagger+m=n+m$. In this case $n=0,1$. Thus $H=j(j+1)-m(m+1)$, with $j\geq|m+n|$ and $n=0,1$. The corresponding spectrum is shown in Fig. 2. The ground state is annihilated by Q and the spectrum is manifestly supersymmetric.

Finally we turn to the case of real interest to us, namely, where $m\neq 0$ and $N=1$ supersymmetry. We find, following the same procedure as before,

$$\{Q^a,Q^{\dagger b}\}=\delta^{ab}H+2m[\delta^{ab}\psi^\dagger\psi-\psi^{\dagger b}\psi^a], \tag{9a}$$

$$[Q^a,H]=0, \tag{9b}$$

$$[H,\psi^\dagger\psi]=[H,\psi^{\dagger b}\psi^a]=0, \tag{9c}$$

$$[Q^b,\psi^\dagger\psi]=Q^b,\quad [Q^b,\psi^{\dagger a}\psi^c]=\delta^{ba}Q^c, \tag{9d}$$

$$[\tfrac{1}{2}\psi^\dagger\sigma^i\psi,\tfrac{1}{2}\psi^\dagger\sigma^j\psi]=\tfrac{1}{2}i\epsilon^{ijk}\psi^\dagger\sigma^k\psi. \tag{9e}$$

Here we have defined H to be that part of the right-hand side of (9a) that commutes with Q. Alternatively, we can start with the Hamiltonian for the $m=0$ case (i.e., no WZ term) and add to it (with a minus sign) the extra piece $m\psi^\dagger\psi$ that had appeared in the Lagrangean (3) due to the addition of the WZ term. Both of these procedures give the same expression for the Hamiltonian:

$$H=(P\bar{P}+\bar{P}P)/2-\tfrac{1}{4}(\psi^\dagger\sigma^i_D\psi)^2-\tfrac{3}{2}\psi^\dagger\psi-m\psi^\dagger\psi-m(m+1)=J^2-m(m+1), \tag{10}$$

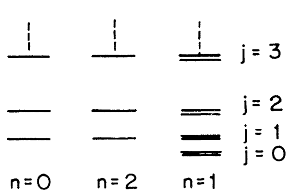

FIG. 1. $N=1$ supersymmetry and $m=0$.

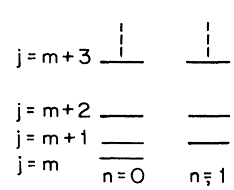

FIG. 2. $N=\frac{1}{2}$ supersymmetry and $m\neq 0$.

with J^i as in the $m=0$ case and satisfying $Z\sigma^i\bar{Z}J^i = \psi\psi^\dagger + m - 1$. Thus we get $H = j(j+1) - m(m+1)$, with $j \geqslant |n+m-1|$. Thus for $n=0$, $j \geqslant |m-1|$ and $H \geqslant -2m$,[15] and for $n=1$, $j \geqslant |m|$ and $H \geqslant 0$. For $n=2$, $j \geqslant |m+1|$ and $H \geqslant 2m+2$. This is shown in Fig. 3. As announced in the introduction both the ground and first excited energy levels have unequal numbers of Bose and Fermi states.

I explain now why the spectrum in Fig. 3 is compatible with supersymmetry. Properties of the spectrum of a Hamiltonian are to be deduced from the algebra obeyed by its symmetry generators. In the present case this is given by Eqs. (9a)–(9e). Relation (9b) requires in the usual way that states related by Q^a have the same energy—this is certainly true of the spectrum shown in Fig 3. The crucial difference is in Eq. (9a) where the right-hand side contains an extra piece in addition to the usual Hamiltonian. These extra terms are the generators of an SU(2) ⊗ U(1) acting on the "Dirac" index a, as seen from Eq. (9d).[16] Using (9a) we can calculate $\{Q^1, Q^{1\dagger}\}$ and $\{Q^2, Q^{2\dagger}\}$ acting on the ground state $|0,0\rangle$ and we find that it is zero for any value of m. Explicitly,

$$\begin{aligned} \{Q^1, Q^{1\dagger}\}|0,0\rangle &= (H + 2m\psi^{2\dagger}\psi^2)|0,0\rangle = 0, \\ \{Q^2, Q^{2\dagger}\}|0,0\rangle &= (H + 2m\psi^{1\dagger}\psi^1)|0,0\rangle = 0. \end{aligned} \tag{11}$$

We further find that

$$\begin{aligned} \{Q^1, Q^{1\dagger}\}|0,1\rangle &= 0, \\ \{Q^1, Q^{1\dagger}\}|1,0\rangle &= 2m|1,0\rangle, \end{aligned} \tag{12}$$

where the states $|0,1\rangle$ and $|1,0\rangle$ are as defined earlier. From (12) we see that one of the supersymmetry charges Q^1 annihilates $|0,1\rangle$ but not $|1,0\rangle$. It thus pairs $|0,0\rangle$ and $|1,0\rangle$ into a supermultiplet but leaves $|0,1\rangle$ a singlet. Similarly Q^2 annihilates $|0,1\rangle$ but not $|1,0\rangle$ and pairs it with $|0,0\rangle$. $|1,0\rangle$ and $|0,1\rangle$ form a doublet under the SU(2) group. We thus see that the first excited level with its two fermionic states $|0,1\rangle$ and $|1,0\rangle$ and the bosonic state $|0,0\rangle$ do form a representation of the superalgebra (9). The rest of the states form supermultiplets in the usual manner. Thus we have a situation where the Hamiltonian is supersymmetric, yet both the ground and first excited levels have different numbers of Bose and Fermi states.

If this phenomenon can be generalized to field theories, it would be of interest, since, as mentioned earlier, it would open up the possibility of constructing realistic theories with supersymmetry breaking perhaps playing quite a different role than the usual one of splitting bosons from fermions. Thus, if for instance expression (3) could be derived as the Lagrangean defining the dynamics of the collective coordinates of a soliton in an underlying supersymmetric field theory, then Fig. 3 would reveal a spectrum of soliton states

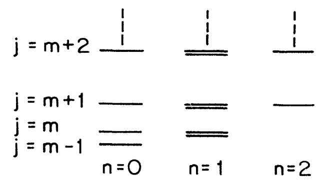

FIG. 3. $N=1$ supersymmetry and $m \neq 0$.

that naively does not appear to be supersymmetric. On a more formal level it would also be interesting to find the generalization of the Witten index for theories of this type.

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[16]This is actually the SU(2/1) superunitary algebra. This phenomenon, where the anticommutator of two supercharges gives an internal symmetry generator, occurs in the gauged supergravity theories. See also V. Rittenberg and S. Yankielowicz, University of Bonn Report No. BONN-HE-84-16, July 1984 (to be published).

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