Chaos and Husimi Distribution Function in Quantum Mechanics

Kin'ya Takahashi and Nobuhiko Saitô
Department of Applied Physics, Waseda University, Okubo, Tokyo 160, Japan
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We study the correspondence between quantum and classical dynamics with Wigner and Husimi distribution functions, respectively. The nature of quantum-mechanical regular and irregular behaviors is discussed with the use of Wigner and Husimi distribution functions. By numerical studies of the chaotic state, the Husimi distribution function is found to be a better representation than the Wigner distribution function, because coarse graining is usually involved implicitly in an observational process using the Wigner distribution function.

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If the localized quantum wave packet does not extend to macroscopic scale in an observational interval, it behaves as a classical point particle according to the Ehrenfest theorem.[1] However, since the Planck constant is finite, the wave packet has a finite width $\Delta q\,\Delta P \sim \hbar$ and consequently it differs more or less significantly from the orbit of a classical point particle. In this paper we study the time development of the quantum wave packet in regular and irregular cases.

To study the correspondence between quantum and classical dynamics, the Wigner distribution function is frequently employed:

$$\rho_{\mathrm{W}}(q,p,t)=(1/2\pi\hbar)\int d\eta\langle q-\tfrac{1}{2}\eta|\psi\rangle\langle\psi|q+\tfrac{1}{2}\eta\rangle\exp(ip\eta/\hbar). \tag{1}$$

However, since ρ_{W} takes negative as well as positive values, ρ_{W} is not exactly a probability distribution function. Gaussian smoothing[2,3] yields a sort of coarse-grained Wigner distribution function that is nonnegative. The class of Gaussian smoothing introduced by Rajagopal[3] is useful because it is related to the coherent state of quantum optics:

$$\bar{\rho}(\langle q\rangle,\langle p\rangle,t)=\int W(q,p,\langle q\rangle,\langle p\rangle)\rho_{\mathrm{W}}(q,p,t)\,dq\,dp, \tag{2}$$

where

$$W(q,p,\langle q\rangle,\langle p\rangle)=(1/\pi\hbar)\exp[-(|\zeta_2|^2/\hbar)(p-\langle p\rangle)^2-(|\zeta_1|^2/\hbar)(q-\langle q\rangle) \\ -(2/\hbar)(|\zeta_1|^2|\zeta_2|^2-1)^{1/2}(p-\langle p\rangle)(q-\langle q\rangle)], \tag{3}$$

and ζ_1 and ζ_2 are two complex numbers with the condition $\zeta_1\zeta_2^*+\zeta_1^*\zeta_2=2$. Now, we take $\zeta_1=\zeta_2=1$ and $\Delta q=\Delta p=(\hbar/2)^{1/2}$, and obtain a Husimi distribution function as[4-6]

$$\rho_{\mathrm{H}}=(2\pi\hbar)^{-1}|\langle\phi_{\langle q\rangle\langle p\rangle}|\psi\rangle|^2, \tag{4}$$

where $\phi_{\langle q\rangle\langle p\rangle}$ is the minimum wave packet,

$$\phi_{\langle q\rangle\langle p\rangle}(x)=\frac{1}{[2\pi(\Delta q)^2]^{1/4}}\exp\left\{\frac{i}{\hbar}\langle p\rangle x-\frac{(x-\langle q\rangle)^2}{4(\Delta q)^2}\right\}. \tag{5}$$

This choice for ζ_1 and ζ_2 enables us to derive simple forms for products and commutation relations of dynamic variables and to develop quantum mechanics quite easily in the Husimi representation.[7]

Now we consider the following system:

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}a_1q^2+\tfrac{1}{4}a_2(1+a_3\sin\omega t)q^4. \tag{6}$$

We take $a_1=1.0$, $a_2=1.0$, $a_3=0.25$, $\omega=0.7$, and $\hbar=0.08$ for the regular case, and $a_1=-1.0$, $a_2=0.25$,

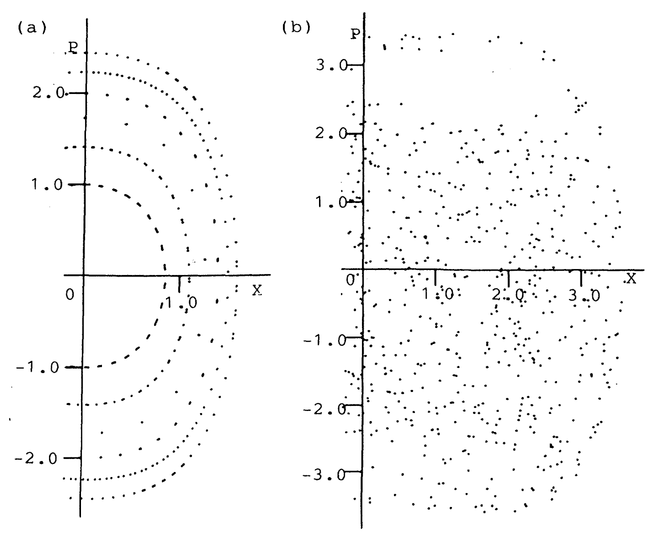

FIG. 1. Classical stroboscopic maps. Since these figures are point symmetric with respect to the origin, the left-hand sides are omitted. (a) Regular case; (b) irregular case.

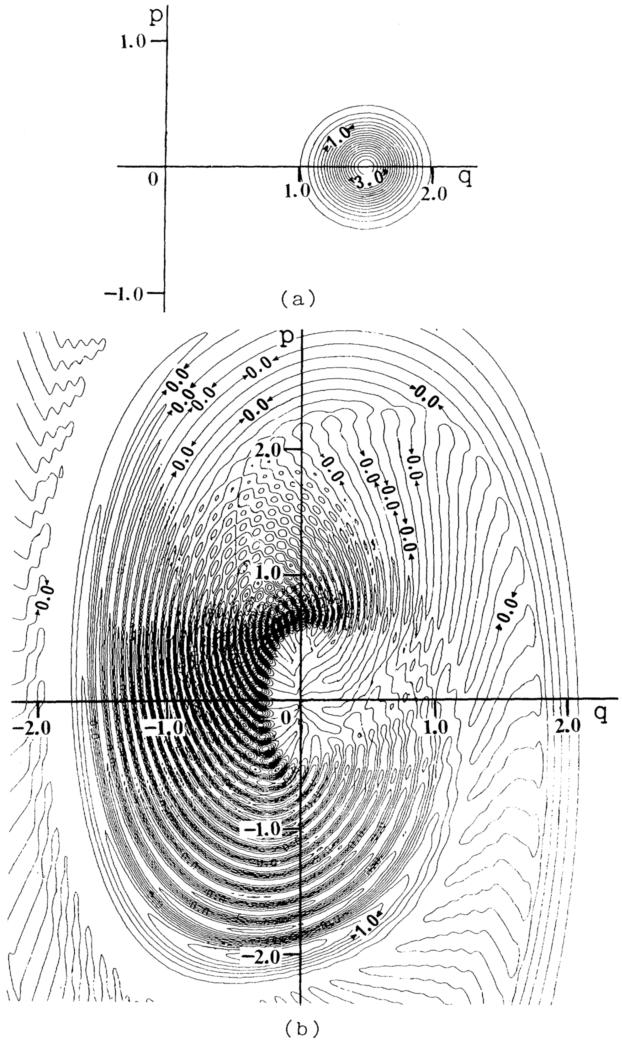

FIG. 2. Contour lines of the Wigner distribution functions for the regular case. (a) $t=0$; (b) $t=T$.

$a_3=0.4$, $\omega=0.7$, and $\hbar=0.04$ for the irregular case. To study the correspondence between classical and quantum behavior, the Planck constant is chosen as $\hbar=0.08$ and 0.04 for regular and irregular cases, respectively. Particularly, the behavior of the irregular case is more complicated than the regular behavior, and then $\hbar$ is chosen as the smaller value ($\hbar=0.04$).

Figures 1(a) and 1(b) show classical stroboscopic maps in the regular and irregular cases, respectively. First, in the regular case, we take as the initial condition a minimum wave packet with center at $\langle q\rangle_0=1.5$ and $\langle p\rangle_0=0$. Figures 2(a) and 3(a) show Wigner and Husimi initial distribution functions, respectively. Figures 2(b) and 3(b) show the Wigner and Husimi distribution functions at $t=T$, respectively, where T is the period of the external force. We find, when compared with Fig. 1(a), that ρ_W spreads all over the inside of

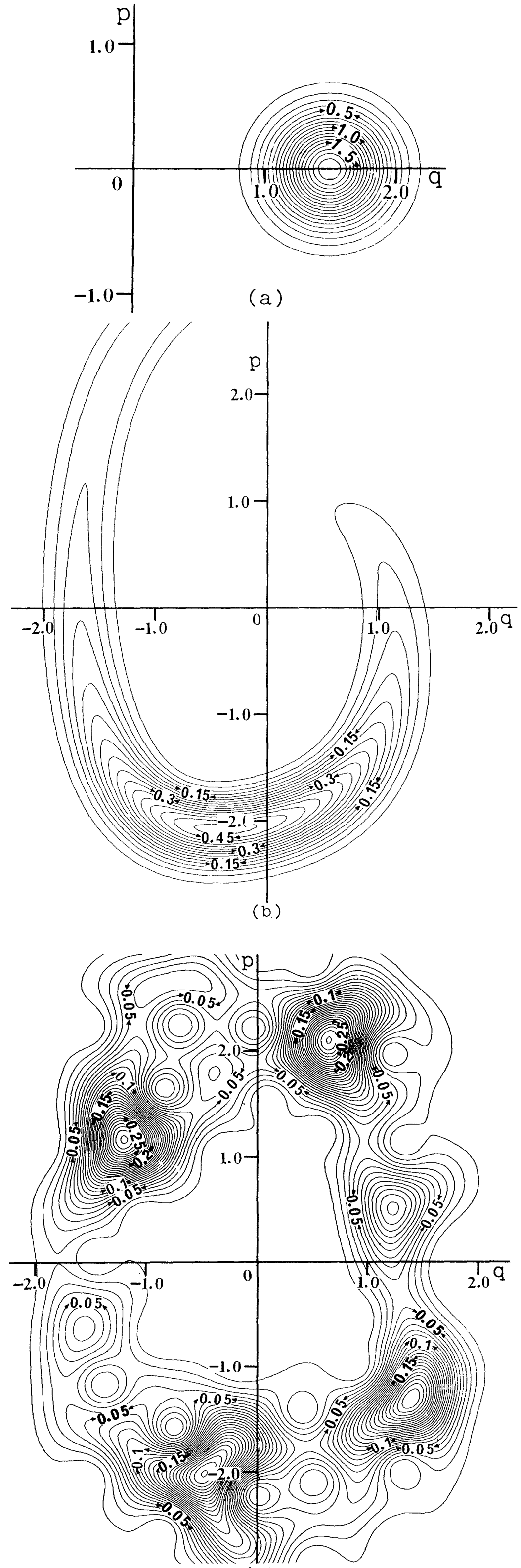

FIG. 3. Contour lines of the Husimi distribution functions for the regular case. (a) $t=0$; (b) $t=T$; (c) $t=10T$.

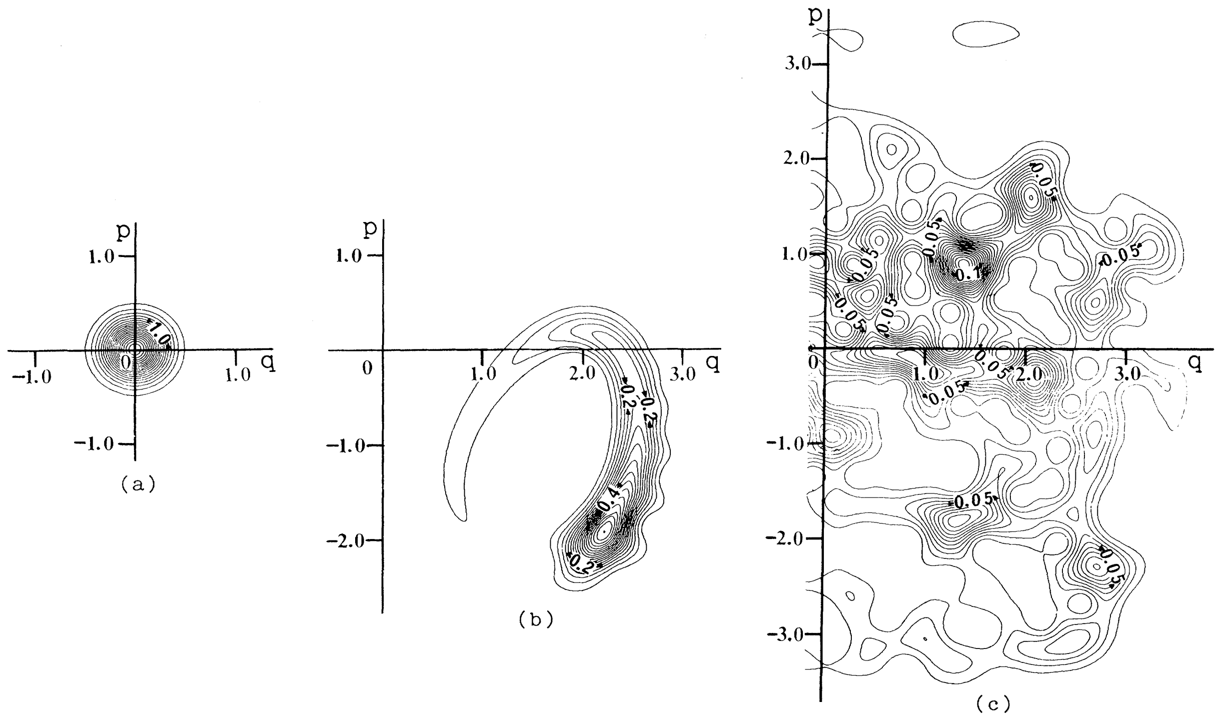

FIG. 4. Contour lines of the Husimi distribution functions for the irregular case. Since these figures are point symmetric with respect to the origin, the left-hand sides of (b) and (c) are omitted. (a) $t=0$; (b) $t=T$; (c) $t=10T$.

the classical torus, while ρ_H, on the other hand, is confined almost completely on the classical torus. Consequently, considering the correspondence between quantum and classical mechanics, the Husimi distribution function is a better representation than the Wigner distribution function, because coarse graining is usually involved in an observational process. Figure 3(c) shows the Husimi distribution function at $t=10T$, which still remains almost completely on the classical torus in the form of a number of small packets. In the irregular case, we take a minimum wave packet with center at the origin as the initial condition. Figures 4(a)–4(c) show the Husimi distribution functions at $t=0$, T, and $10T$, respectively.

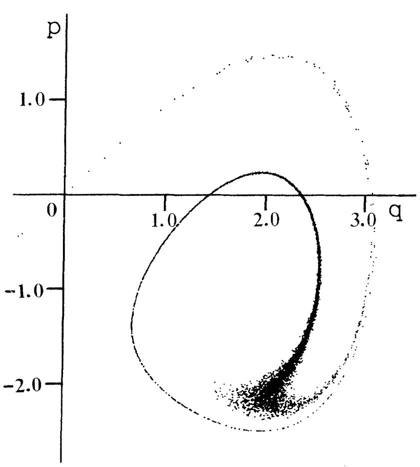

FIG. 5. Time development of ten thousand classical point particles at $t=T$. Since the distribution of these particles is point symmetric with respect to the origin, the left-hand side is omitted.

To make a comparison with the classical case, we calculated the orbits of ten thousand classical particles whose initial conditions are assumed to obey the same probability distribution as $\rho_H(\langle q\rangle,\langle p\rangle,0)$. Figure 5 shows them at $t=T$. At $t=T$, ρ_H coincides quite well with the distribution of classical particles. At $t=10T$, ρ_H extends to nearly all the classical irregular regions, but it is distributed inhomogeneously and rather partially localized on small wave packets. We find that these small quantum wave packets cannot easily be destroyed, but tend to localize, in contrast to the classical case where no localization can be found. These small packets are considered as an interference effect of wave functions. More detailed studies on this subject will be reported elsewhere, together with theoretical studies on the dynamics and eigenvalue problems in the Husimi distribution function.[7]

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