Approach *à la* Borland to Multidimensional Localization

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We develop for the first time an approach *à la* Borland to Anderson localization in multidimensional systems; it provides a proof of localization when the Green's function decays exponentially, e.g., at large disorder or large energy. This approach also provides results about the Lyapunov exponents associated with a quasi-one-dimensional system. Finally we obtain the result that the singular continuous spectrum, found in some incommensurate systems, turns into exponential localization under arbitrarily small local perturbations.

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The problem of localization of electrons in disordered solids has been of theoretical and experimental interest for years among condensed matter physicists; for reviews on this topic we refer to Thouless,[1] Bergman,[2] and Souillard.[3] In this paper we consider the Anderson model of electrons in crystals with impurities; it is a tight-binding model associated with a d-dimensional lattice or with a restriction of it, such as a wire (infinite cylinder). The Hamiltonian of the electron is given by

$$(H\Psi)(x)=\sum_{|x-y|=1}\Psi(y)+V(x)\Psi(x), \qquad (1)$$

where the $V(x)$'s are independent random variables, that we shall suppose, for simplicity, are identically distributed. In particular, we want to study the stationary states, which satisfy

$$H\Psi=E\Psi. \qquad (2)$$

It was first predicted by Anderson[4] in 1958 that for a typical sample all states should be exponentially localized for large enough disorder and that for any disorder the states should be localized at low enough energy, with a transition to extended states expected to occur for weaker disorder and in the middle of the band. It was later shown by Mott and Twose[5] that in one dimension all states are exponentially localized for arbitrarily weak disorder. More recent scaling arguments[6] have yielded the result that all states should be exponentially localized in two dimensions, again for arbitrarily weak disorder.

For one-dimensional (1-D) systems, a fundamental step was achieved by Borland in 1963.[7] In his well-known work, he gave a particularly simple approach to 1-D localization which furthermore shows that the localization length is the inverse of the Lyapunov exponent associated to some product of 2×2 random matrices which is particularly simple to compute numerically.

In the present paper we want to develop for the first time an approach *à la* Borland to multidimensional systems. It gives a simple approach to Anderson localization in the case when the Green's function decays exponentially, a property, e.g., of sufficiently disordered systems. In order to realize this program we will first have to clarify one difficulty intrinsic to Borland's approach and connected with some known counterexample and then to give the conditions under which Borland's approach is correct; this step is achieved by use of a new idea of Kotani,[8] also contained implicitly in the work of Carmona.[9] From this discussion we will see that our approach to multidimensional localization, in fact, gives a mathematically rigorous proof of it! But we will also "get for free" some by-products: In particular, we find that arbitrarily small local perturbations of the potential at two sites make the "singular continuous spectrum" (sometimes called "exotic" states) obtained in some classes of one-dimensional incommensurate systems disappear and instead all eigenstates become exponentially localized! We also prove that the smallest of the Lyapunov exponents associated to the Anderson model in a strip, which is used in finite-size scaling arguments, does not vanish in the limit of infinitely large strips, e.g., at large enough disorder, a result that is physically intuitive but an old mathematical conjecture.

Borland's approach can be summarized in the following way. Consider a 1-D tight-binding model for which Eq. (2) reads

$$\Psi_{n+1}=-\Psi_{n-1}+(E-V_n)\Psi_n, \qquad (3)$$

or, with use of a transfer matrix formalism

$$(\Psi_{n+1},\Psi_n)=M_n(\Psi_n,\Psi_{n-1}) = M_nM_{n-1}\cdots M_1(\Psi_1,\Psi_0), \qquad (4)$$

where M_n is a 2×2 random transfer matrix. Borland shows that for any typical sample (i.e., for almost every sequence V_n) the norm of the product $\prod M_i$ of random matrices increases exponentially with n, so that (noticing that $\det M_n=1$) all solutions of Eq. (3) are exponentially increasing at $+\infty$ except one which decays exponentially at $+\infty$, and similarly at $-\infty$. Borland then argues that since physical states can grow exponentially neither at $+\infty$, nor at $-\infty$, the only physical energies are those E for which the unique solution exponentially decreasing at $+\infty$ coincides with the unique solution for exponentially decreasing at $-\infty$, and as a consequence the only physical states are exponentially localized states. Furthermore, this argument clearly identifies the inverse localization length as the rate of exponential growth of the product of random matrices, which is the so-called Lyapunov exponent and which is particularly easy to compute numerically.

Actually there is a difficulty with this very appealing argument as can be seen, for instance, in the following tight-binding model for an incommensurate system:

$$\Psi_{n+1}+\Psi_{n-1}+\lambda\cos[2\pi(\alpha n+\theta)]\Psi_n=E\Psi_n. \quad (5)$$

It is known that for α irrational and $\lambda>2$ the Lyapunov exponent is strictly positive but if α is an irrational number "well approximated by rationals" it is known[10] that for almost every θ there is no localized state! We thus want to understand this paradoxal situation and clarify under which conditions, if any, Borland's argument is valid.

So what is the difficulty in Borland's method? Let us denote by Ω the set of the potentials $V=\{V_n\}_{n\in Z}$; V is chosen in Ω according to some probability distribution P. The exponential growth of the product of transfer matrices is known to be true, an energy E being given, only for all potentials outside of some set Ω_E of zero probability, i.e., $P(\Omega_E)=0$. In fact, what we would like to do is to first choose the potential at random and then for this potential get results valid for every energy, or at least all physically relevant energies (more on this later). However, the exponential-growth property cannot be shown directly to be true, V being given, for every energy because Ω_E does depend on the energy; in fact for any given potential the exponential behavior holds only outside of some (dense) set of energies $S(V)$ of zero measure, which could happen to contain physically relevant values of the energy for which we could have "exotic states."

Let us now make more precise what we mean by a physically relevant energy and a physical state: What we want, in fact, is to discuss the time-dependent properties of solutions of the time-dependent Schrödinger equation with a given square integrable initial condition. This time evolution can be computed if we know the solutions of the stationary Schrödinger equation (2), but in fact we do not need to know all the solutions of (2) for all E but only for some of them. More precisely we need to consider only those E (the physically relevant energies) for which there exist nonexponentially increasing solutions, and only those Ψ which are not exponentially increasing. Technically speaking the relevant energies E are those for which there exist (polynomially bounded) generalized eigenfunctions Ψ solutions of (2), and it is known that those of E are of full spectral measure. Note that if H has an eigenvalue spectrum, the relevant energies are the eigenvalues.

In order to study the exceptional sets $S(V)$, we first notice that the exponential behavior of $\prod M_i$ is the asymptotic property and thus does not depend on the values of $V(0)$. Then the exceptional set of energies $S(V)$ is constant under changes of $V(0)$. On the other hand, the physically relevant energies can be obtained from the eigenvalues of the restriction H_Λ of H to large boxes Λ (in mathematical terms the spectral measures of H_Λ weakly converge to the spectral measure of H as $\Lambda\nearrow Z$). The eigenvalues of H_Λ vary smoothly and monotonically if $V(0)$ is varied, and thus the experimental set of energies $S(V)$ cannot intersect the set of eigenvalues of H_Λ except for an exceptional set of values of the potential at 0; by the convergence of the spectral measures this is enough to ensure that, for P-almost-every V, $S(V)$ does not contain relevant energies. This argument takes its inspiration from Kotani,[8] where Kotani uses the boundary condition as a parameter [like our $V(0)$] in the semi-infinite system.

This suggests the natural condition under which Borland's argument can be applied: It is sufficient[11] that the potentials $V(0)$ and $V(1)$ be distributed with a continuous density, however small their range of values [the variation of $V(1)$ in addition to $V(0)$ is needed in the above perturbation argument to deal with the cases when $\Psi(0)=0$], to conclude that the matrix product grows exponentially, for almost any configuration of V, and for the relevant energies of H. Thus the solutions of (3) behave exponentially at both infinities; as the generalized eigenfunctions cannot grow exponentially they decay exponentially at both $\pm\infty$. This proves that the generalized eigenfunctions are exponentially localized eigenstates and we see that the inverse localization length is identified as the exponential growth rate of $\prod M_i$, i.e., the Lyapunov exponent.

The above provides us with a simple proof of localization in disordered 1-D systems,[11] but the result is not a surprise. A consequence which may be more surprising is the following: Consider a tight-binding model of incommensurate systems such as the one discussed above for which there is a positive Lyapunov exponent but no localized state; arbitrarily small random perturbations on the potential, e.g., at sites 0 and 1 make the singular continuous spectrum disappear and give rise to exponential localization of all states with the Lyapunov exponent as the inverse localization length! Thus the singular-continuous-spectrum feature is irrelevant to physics in these cases.

A straightforward generalization[11] gives the same results for the case of a wire or quasi-one-dimensional system, where one considers an infinite cylinder of given finite section. If the section has N sites, the transfer matrices are now $2N\times 2N$ matrices. In this case, the localization length is found to be the inverse of the smallest positive Lyapunov exponent of the product of transfer matrices; still, localization occurs for arbitrarily small disorder, under the hypothesis that the potentials are independently distributed with a density with respect to the Lebesque measure.

Borland's approach can, in fact, be extended to multidimensional systems as we are now in a position to show. We would like to argue that any solution of (2) either decays or grows exponentially, as soon as the disorder is large enough or when one considers states of sufficiently low energy. Unfortunately in multidimensional systems, the previously considered transfer matrices become infinite, and are not directly usable. The behavior of the solutions is, in fact, ruled by the behavior of the Green's function, as shown by Delyon, Lévy, and Souillard[12] and previously by Martinelli and Scoppola.[13] In particular, a nice sufficient condition to get the expected exponential behavior, increasing or decaying, of the formal solutions of (2) in any direction is that the Green's function decays exponentially at infinity. Indeed, suppose that the Green's function of the infinite multidimensional system decays exponentially; namely, suppose that we have the following bound:

$$G(x,y;E) < C(x,E,V)\exp\{-\alpha(E)|x-y|\}, \qquad (6)$$

with $\alpha(E)>0$. This allows us to give a sense, column by column, to the operator product GH, so that, applying G to the left of the equation

$$(H+\delta V|0\rangle\langle 0|)\Psi = E\Psi, \qquad (7)$$

we get the result that, if Ψ satisfies (7) and does not increase exponentially at infinity, then it is proportional to $G|0\rangle$ and exists only if δV is equal to $1/G(0,0;E)$. Now we argue as Borland does that the physically relevant states cannot grow exponentially at infinity so that, by (6), the eigenstates of H necessarily decay exponentially.

Still in this *à la* Borland multidimensional method we assume that the distributions of the potentials have densities; this still allows us to let the potential vary locally in a continuous way and, as in the 1-D case, discard the exceptional situations. This proves that, at any dimension and under the assumption that the random potential has a density, one has only exponentially localized states as soon as the Green's function decays exponentially. This condition, or at least some analogous condition on finite systems which is sufficient for our argument, comes from a control of the resonances of the system and was proven by Fröhlich and Spencer[14] (see also Holden and Martinelli[15]) at sufficiently high disorder or for low enough energy. Thus our approach provides us with the first mathematical proof that, as predicted by Anderson, all the eigenstates are exponentially localized in these regimes.

Whereas results are hardly available directly on the behavior of the smallest Lyapunov exponent considered in the study of the wire when the section increases, one consequence of our above results is that this Lyapunov exponent—e.g., at large disorder—remains bounded below, as the section of the cylinder increases. This fact is not surprising physically since this Lyapunov exponent should go to the inverse localization length which is nonzero in this regime, but, on the mathematical side, this used to be a conjecture difficult to attack directly.

As a summary, we have discussed in detail the classical Borland argument. We have clarified the conditions under which this argument can be applied. We then developed an approach *à la* Borland to localization in multidimensional systems. The conjunction of these two results yields the first proof of Anderson localization for multidimensional systems at large disorder or at low energy. We obtained in passing the proof that the smallest Lyapunov exponent in a wire does not go to zero for infinite width, e.g., at large disorder. Another consequence of our work is that the known cases where the spectrum is found to be singular continuous, i.e., the systems with quasiperiodic potential described above, must be considered as exceptional and physically irrelevant in the sense that they are unstable under perturbations of some isolated potential, as any such perturbation leads to exponential localization.

While we were completing this work, we learned that Fröhlich *et al.*[16] and Simon, Taylor, and Wolff[17] have reached similar conclusions as far as a proof of localization is concerned. In the first reference one will find a preliminary version of a different proof. The second one is an announcement of a proof similar to the one of Ref. 12, with some technical differences, but with no connection to Borland's approach.

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