

Measuring Maxwell's Displacement Current Inside a Capacitor

D. F. Bartlett and T. R. Corle^(a)

Department of Physics, University of Colorado, Boulder, Colorado 80309

(Received 25 February 1985)

We have measured the magnetic field directly inside a thin, circular, parallel-plate capacitor as it is being charged. We find that this field varies linearly with distance from the axis, as is to be expected if a uniform displacement current flows between the plates. The measured slope of B vs r agrees with predictions to within 5%.

PACS numbers: 41.10.Fs, 03.50.De, 06.30.Lz

A century ago Maxwell wrote, "One of the chief peculiarities of this thesis is the doctrine which asserts that the true electric current, that upon which electromagnetic phenomena depend, is not the same thing as the current of conduction but that the time derivative of the electric displacement must be taken into account."¹ The existence of the displacement current was shown dramatically by Hertz in a series of experiments in which electromagnetic waves were generated and detected.² To our knowledge, however, no one has as yet measured the displacement current in the apparently direct fashion by observing the magnetic field inside a capacitor that is being slowly charged.³ That there should be such a current is shown in most texts.⁴

Imagine that a circular, parallel-plate capacitor is being charged by a conduction current I_C . By Ampère's law, the integral of B around the loop shown in Fig. 1 is $4\pi I_{\text{enc}}/c$. If measured by the current piercing the surface S_1 , the current enclosed is simply I_C . However, no conduction current pierces surface S_2 ; thus to ensure the uniqueness of Ampère's law we must hypothesize a displacement current I_D inside the capacitor. Additionally, in the central region the current density $\mathbf{J}_D = (1/4\pi)(\partial\mathbf{D}/\partial t)$ with $J_D = (1/4\pi d)(dvV/dt)$ is a constant and, by Ampère's law, \mathbf{B} is always in

the azimuthal direction and has a magnitude $B = (r/2cd)/(dV/dt)$. Here d and V are the separation and potential difference between the plates. Hence B increases linearly with r with a slope $dB/dr = (1/2cd)(dV/dt)$ up to the edge of the plates. From the edge outwards B increases as $1/r$.

Ideally the experiment should be done with a direct current. However, the plates would quickly become so highly charged that they would spark. A simple expedient is to charge the plates quasistatically with an audio frequency ω . If $V = V_0 \sin \omega t$, the magnetic field is $B = (r\omega V_0/2cd)\cos \omega t$. In the case of a hand-sized capacitor charged to a kilovolt and with plates separated by a centimeter the maximum value of B is only about half a microgauss.

To detect this small field we use a superconducting quantum interference detector (SQUID). The detector consists of a commercial probe⁵ coupled by short twisted leads to a 1.5-mm-diam, 25-turn coil. The entire probe may be moved vertically between the plates, thus sampling B_ϕ as a function of r . The coil is shielded from electric fields by being enclosed in a stainless steel tube. The Josephson junction of the probe is shielded from both stray electric and magnetic fields by a superconducting lead shield. (See Fig. 2.)

To maintain the low temperatures needed for superconductivity, the capacitor is immersed in liquid helium inside a 1-m-deep, 13-cm-diam bucket Dewar. Unfortunately, all access to the Dewar is from the top. Although the axis of the capacitor is horizontal, it is fed by currents flowing through two vertical wires, thus compromising the ideal arrangement of Fig. 1. The plates and connecting wires are driven at 0.8 sV (240 V_{rms}) by two 1250-Hz signals that are 180° out of phase. The driving signals are derived from a single audio oscillator driving an amplifier, whose center-tapped output is further amplified by two automobile ignition coils. This "push-pull" arrangement ensures that the grounded stainless steel tube of the magnetometer will not appreciably disturb the electric fields inside the capacitor. To detect the field, the signal from the SQUID magnetometer is amplified and compared with that from the audio oscillator in a phase-sensitive or "lock-in" amplifier.

We use an *in situ* method to set the phase of the

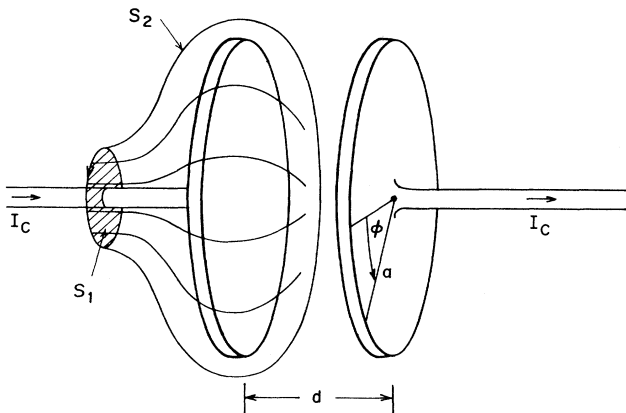


FIG. 1. Ampère's law and Maxwell's displacement current.

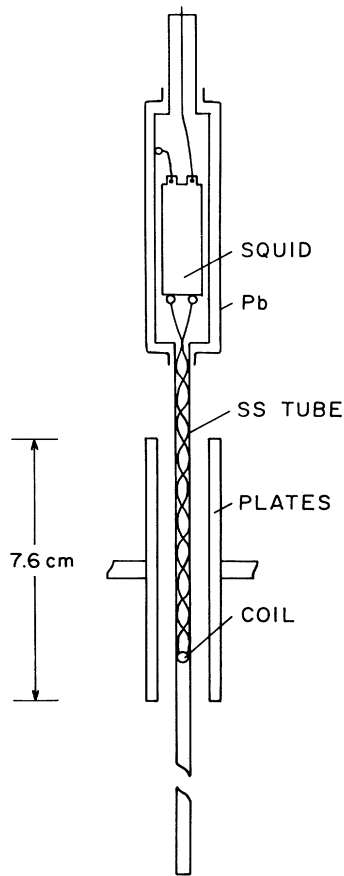


FIG. 2. SQUID magnetometer. The permeability of the nonmagnetic stainless steel tube is unity for the small fields used in this experiment; thus this tube shields electric, not magnetic, fields.

lock-in detector to agree with that of I_D . The vertical leads feeding the capacitor are symmetric about the axis of the Dewar and are separated by only 4 cm. The current in these leads is in phase with the displacement current. By raising the SQUID probe into the region above the capacitor we sample the magnetic field of the lock-in detector. To minimize the unwanted influence of the vertical charging currents on the magnetic field within the plates, the leads are bent through 90° before making their final horizontal, axial approach to the capacitor (see Fig. 3). By rotating the SQUID probe through 90° we can measure either the magnetic field of the displacement current B_D or the magnetic field of the vertical conduction currents B_C .

This arrangement leaves us with some problems. Current in the lower 90° bends shown in Fig. 3 gives an unwanted contribution to the measured B_D . Conversely, current in the two axial leads, which should stretch to infinity, can only contribute for a short distance. This situation would lead to a large but calcul-

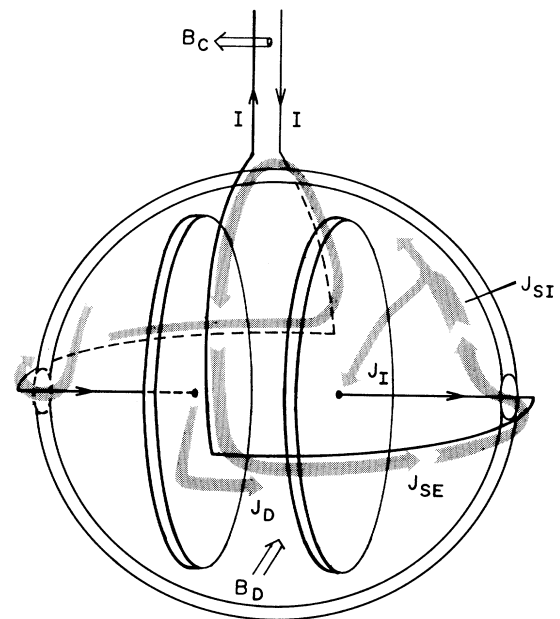


FIG. 3. Apparatus. Capacitor within superconducting sphere. The light, broad arrows show schematically the various current densities. See text for definitions of symbols.

able correction to the expected field were it not for a more insidious problem. The stray capacitance of the lead wires to the walls of the Dewar is several times the capacity between the plates. Thus much of the current which enters the Dewar does not travel through the plates but rather travels (as displacement current) to the walls of the Dewar. From there the current is conducted in a highly asymmetric manner through welded seams to the other side of the Dewar. Here it travels again as displacement current to the other wire and returns to the amplifier. If unshielded these unwanted currents produce an uncalculable spurious B_D comparable to the desired signal.

Fortunately, these currents can be shielded by use of the Meissner effect.⁶ Surrounding the plates is a 12.5-cm-diam copper spherical shell which has been coated internally with 50–50 Pb-Sn solder. This solder is superconducting and so, when put in a magnetic field, maintains $\mathbf{B}=0$ inside the solder itself. Currents external to the shield cause superficial currents $\mathbf{J}_{SE}(\theta, \phi)$ to flow in both the polar direction $\hat{\theta}$ and the azimuthal direction $\hat{\phi}$ on the outside of the solder shell. The combined effect of \mathbf{J}_{SE} and all external currents is to maintain $\mathbf{B}=0$ in the solder material which will (by the uniqueness theorem) also maintain $\mathbf{B}=0$ at all internal points.

The superconducting shell also shields against unwanted internal \mathbf{J}_I arising from displacement current traveling to the shell from the plates and

the horizontal wires. In this case the argument is a little subtle. The internal currents, both the unwanted \mathbf{J}_I and the essential \mathbf{J}_D between the plates, are azimuthally symmetric; they have no ϕ component. (See Fig. 3.) Even so, were the shell not superconducting it could easily support nonazimuthally symmetric conduction currents and we would again have a noncalculable contribution to the measured \mathbf{B} field. But the shell is a superconductor. To maintain $\mathbf{B}=0$ inside the solder material, an azimuthally symmetric current $\mathbf{J}_{SI}(\theta)$ must flow on the inside surface to balance \mathbf{J}_I and \mathbf{J}_D .

Since all the internal currents now have azimuthal symmetry, the vector potential \mathbf{A} must have neither a ϕ component nor be dependent upon ϕ . Thus $\mathbf{B}=\nabla\times\mathbf{A}$ within the shell everywhere must be in the ϕ direction and must be independent of ϕ . The magnetic field can then readily be calculated by Ampère's law with the result that within the shell the predicted field is just the same as it would be if the capacitor were completely isolated, fed by infinitely long axial leads. Thus the azimuthally symmetric superconducting shield not only shields, it makes our cramped physical apparatus effectively the same as the idealized infinite apparatus of Fig. 1.

The results obtained are shown in Fig. 4, where we have plotted B_D versus the height of the probe h . The dots represent data from two sets of measurements. The theoretical curve clearly shows a linear form between the bottom and top of the capacitor plates with a falloff in the region of the fringing fields. This curve, which allows for the finite geometry of the plates and the copper spherical shield, was determined by a numerical relaxation method.⁷ There were two free parameters: the position of the capacitor center line and the slope of B_D vs h in the linear region.

In principle, the depth of the capacitor axis below the top of the Dewar can be accurately measured with a meter stick. But the capacitor is mounted at the end of a long plastic tube. The unknown temperature profile in the Dewar makes it difficult to allow for thermal contractions, and so we feel that the location of the center line is best determined by the data themselves.

The slope which best fits the experimental data is $dB_D/dh=0.171\pm 0.005\ \mu\text{G}/\text{cm}$. The predicted slope is

$$dB/dr = K\omega V/(2cd) = 0.180 \pm 0.005\ \mu\text{G}/\text{cm}.$$

Here K , the dielectric constant of liquid helium, is 1.05, $\omega=7850\ \text{rad/s}$, $V=1.60\pm 0.08\ \text{sV}_{\text{rms}}$, and d , the plate separation, is 1.22 cm. The agreement between measurement and theory is at the level of 5%.

Clearly the time derivative of \mathbf{D} must be "taken into account" if we use Ampère's law. But what if we use the Biot-Savart law? Do we need to consider \mathbf{J}_D as well as \mathbf{J}_C ? Curiously we do not. As French and Tess-

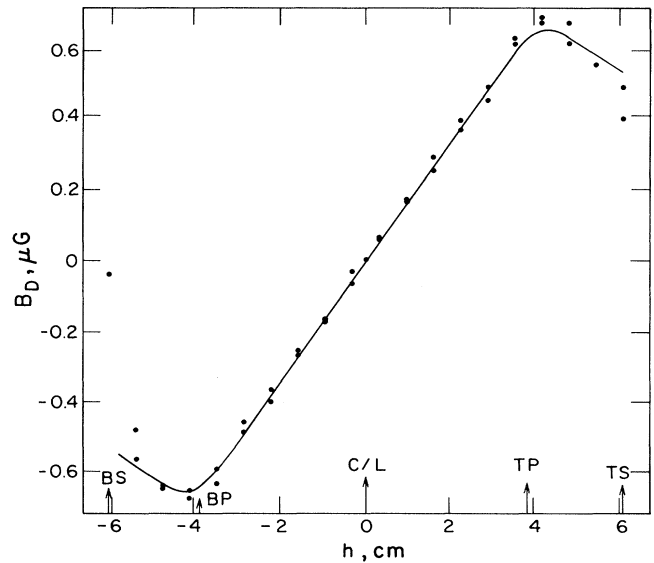


FIG. 4. Results. The scale of B_D was determined in a separate calibration in which the coil was placed inside a small Helmholtz coil carrying a known 1250-Hz current. BS = bottom of sphere, BP = bottom of plates, C/L = center line, TP = top of plates, and TS = top of sphere.

man have shown, in any quasistatic measurement \mathbf{B} can be found from the conduction currents alone. The argument is simple. According to the Biot-Savart law

$$\mathbf{B} = \int \frac{\mathbf{J} \times \hat{\mathbf{R}} d\tau'}{cR^2} = \int \frac{\nabla \times \mathbf{J}}{cR} d\tau',$$

where $\mathbf{J}=\mathbf{J}_D+\mathbf{J}_C$ and R is the vector from the source point (primed) to the field point. But $\nabla \times \mathbf{J}_D \propto \nabla \times \partial \mathbf{E}/\partial t$ and for quasistatic measurements $\nabla \times \mathbf{E} \approx 0$. Thus only \mathbf{J}_C contributes.⁸

What we have shown, then, is that the Biot-Savart law applies to open as well as to closed circuits. One may write the differential form of this law as

$$d\mathbf{B} = I_C d\mathbf{l} \times \hat{\mathbf{R}}/cR^2,$$

without the usual caveat that only the integral around a closed loop is meaningful.

We gratefully acknowledge the help given by Peter Wilcoxon, William Stone, and Tracey Clyde. Very useful comments were received from A. P. French. This work was partially supported by a National Bureau of Standards Precision Measurement Grant funded in cooperation with the National Science Foundation and by a grant from the Research Corporation.

(a)Now at Department of Applied Physics, Stanford

University, Stanford, California 94305.

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