Permanent Quark Confinement in Four-Dimensional Hierarchical Lattice Gauge Theories of Migdal-Kadanoff Type

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Permanent quark confinement is established for four-dimensional hierarchical lattice gauge theories in which the Migdal-Kadanoff approximate renormalization recursion formulas hold exactly. This holds for gauge groups $G=\mathrm{SU}(N)$ as well as $G=\mathrm{U}(N)$.

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In spite of tremendous efforts made by many people, a rigorous proof of permanent quark confinement in four-dimensional (4D) non-Abelian lattice gauge theories is still not in sight. The rigorous real-space renormalization-group method[1] is presumably most promising, but it is incredibly difficult to define block spins for group variables.[2,3]

I recently constructed hierarchical lattice gauge field models[4] where the Migdal-Kadanoff approximate renormalization-group methods[5,6] work precisely, and thus the block-spin transformations have simple closed forms in these systems. These systems may look quite artificial, but surprisingly enough it turned out that they do fairly well as the first approximation.[7] In this Letter, I show that quark confinement is realized in these systems of gauge group $G=\mathrm{SU}(N)$ or $G=\mathrm{U}(N)$. The proof is rather subtle and depends only on the fact that the gauge $G=\mathrm{SU}(N)$ is compact and contains the Cartan subgroups $\mathrm{U}(1)^{N-1}$, and can be analytically continued. Therefore these approximate formulas do not distinguish Abelian groups from non-Abelian ones. I thus believe that it is very important to find a missing link which connects the real systems and these approximate ones. This problem is now under intensive consideration.[8]

The hierarchical lattice gauge theories in four dimensions are made as follows (Ref. 4; see also Griffiths and Kaufman[9] and Collet and Eckmann[9]): (i) Use a kind of temporal gauge. Set $G\ni v_b=1$ for all (vertical) bonds $b=(x,x+e_3)$ and $b=(x,x+e_4)$, where e_μ is the unit vector in the μ direction. (ii) For plaquettes p on the x_1-x_2 planes, there correspond the standard Wilson actions $A_p=\beta\,\mathrm{Re}\,\mathrm{Tr}(v_p-1)$, where $v_p=\prod_{b\in\partial p}v_b$. (iii) If $\beta=0$ for all other plaquettes on x_i-x_j planes with $(i,j)\neq(1,2)$, the system is just a set of 2D lattice gauge theories which are exactly soluble. So we glue them together in a hierarchical way: For plaquettes p on other planes, set $\beta=0$ or $\beta=\infty$ in $\beta\,\mathrm{Re}\,\mathrm{Tr}(v_p-1)$ depending on where they are. [These plaquette actions take the form $\beta\,\mathrm{Re}\,\mathrm{Tr}(v_b v_{b'}^{-1}-1)$, where b and b' are parallel nearest-neighbor bonds contained in two different x_1-x_2 planes. So $\beta=\infty$ means that v_b and $v_{b'}$ are identified.] See Ref. 4 for the details. This construction may be well understood by Fig. 1, which explains how the block-spin transformations are carried out in these hierarchical lattices (of Migdal type), and corresponds to $\lambda=2$ and $D=4$ in Ref. 4.

Thus this system obviously satisfies the following recursion formulas:

$$f^{(n+1)}(v)=\mathscr{N}^{-1}\left(\int\prod_{p\subset\Lambda}f^{(n)}(v_p)\prod_{b\in\Lambda_0}dv_b\right)^{\lambda^2},\quad(1)$$

where Λ is a block plaquette of size $\lambda\times\lambda$ (in the units of λ^n), p are unit plaquettes in Λ (in units of λ^n), $b\in\Lambda_0$ are internal bonds, $\mathscr{N}$ is the constant chosen so that $f^{(n+1)}(1)=1$, and dv is the normalized Haar measure on G. $v=\prod_{b\in\partial\Lambda}v_b$ (ordered along $\partial\Lambda$) plays a role of the block spin, and of course, $f^{(0)}(v)=\exp[\beta\,\mathrm{Re}\,\mathrm{Tr}(v-1)]$ is the starting point. I wish to show that the effective Wilson action $\ln f^{(n)}(v)$ at the distance scale λ^n tends to zero as

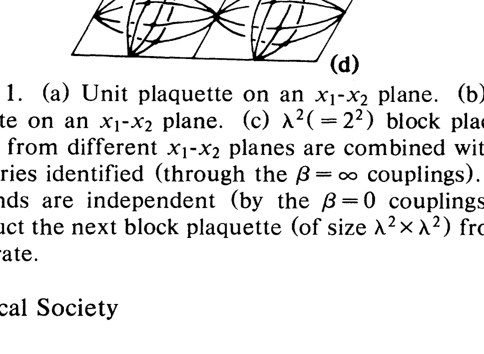

FIG. 1. (a) Unit plaquette on an x_1-x_2 plane. (b) Block plaquette on an x_1-x_2 plane. (c) $\lambda^2(=2^2)$ block plaquettes coming from different x_1-x_2 planes are combined with their boundaries identified (through the $\beta=\infty$ couplings). Internal bonds are independent (by the $\beta=0$ couplings). (d) Construct the next block plaquette (of size $\lambda^2\times\lambda^2$) from (c), and iterate.

$n \to \infty$. By an easy gauge transformation, I have

$$f^{(n+1)}(v) = \mathcal{N}^{-1}\left[\int f^{(n)}(vv_1^{-1}) \cdots f^{(n)}(v_{r-2}v_{r-1}^{-1}) f^{(n)}(v_{r-1}) \prod dv_i\right]^r, \tag{2}$$

where $r = \lambda^2$. This is a well-known Migdal recursion formula in spin systems. The recursion formulas of Kadanoff type are obtained from similar hierarchical lattices; see Refs. 4 and 9. Without loss, I can set $\lambda = \sqrt{2}$ and I have two types of recursion formulas: the Midgal type,

$$f^{(n+1)}(v) = R_{\rm M}(f^{(n)})(v), \quad R_{\rm M}(f)(v) = \left[\int f(vv_1^{-1}) f(v_1)\, dv_1 \Big/ \int f(v_1)^2\, dv_1\right]^2, \tag{3}$$

and the Kadanoff type,

$$f^{(n+1)} = R_{\rm K}(f^{(n)})(v), \quad R_{\rm K}(f)(v) = \int [f(vv_1^{-1}) f(v_1)]^2\, dv_1 \Big/ \int [f(v_1)^2]^2\, dv_1. \tag{4}$$

More generally, $f^{(0)}$ can be chosen from $\mathcal{F} = \{f\}$: (i)

$$1 = f(1) \geq f(v) = f(v^{-1}) \geq 0, \quad f(uv) = f(vu)\ \text{(i.e., class functions)}. \tag{5}$$

(ii) Let $\sigma(z) = \exp(i\sum_{i=1}^{s} z_i\lambda_i)$, $\tau(\omega) = \exp(i\sum_{j=1}^{t}\omega_j\tilde{\lambda}_j)$, where $\{\lambda_i, \tilde{\lambda}_j\}$ are $N\times N$ Hermitian traceless [for $G = \mathrm{SU}(N)$] matrices normalized so that $\|\lambda_i\| = \|\tilde{\lambda}_j\| = 1$. Then there exists a strictly positive constant l such that $f(\sigma(z)v\tau(\omega)\tilde{v})$ is analytic in $D_l = \{(z_i, \omega_j) \in C^{s+t}; |\mathrm{Im} z_i|, |\mathrm{Im}\omega_j| \leq l\}$, and satisfies a bound

$$|f| \leq f(\sigma(\mathrm{Re} z)v\tau(\mathrm{Re}\omega)\tilde{v}) \exp\{\tfrac{1}{2}\beta C(N)[\textstyle\sum(\mathrm{Im} z_i)^2 + \sum(\mathrm{Im}\omega_j)^2]\} \tag{6}$$

uniformly in $v, \tilde{v} \in G$, in the region D_l, where $C(N) = C(N;s,t;l)$ is a positive constant. (iii) $f(v)$ is of positive type (the coefficient of the character expansion are positive).

Obviously $f^{(0)}(v) = \exp[\beta\,\mathrm{Re}\,\mathrm{Tr}(v-1)]$ belongs to F. In fact, the properties (i) and (iii) are easy to see, and as for (ii), note that

$$f^{(0)} = \exp\{\beta/2\,\mathrm{Tr}[\sigma(z)v\tau(\omega)\tilde{v} + \tilde{v}^*\tau(\overline{\omega})^* v^*\sigma(\overline{z})^* - 2]\}, \tag{7}$$

and expand $\sigma(z) = \exp(iA - B)$ with $A = \sum(\mathrm{Re} z_i)\lambda_i$ and $B = \sum(\mathrm{Im} z_i)\lambda_i$ [respectively, $\tau(\omega) = \exp(i\tilde{A} - \tilde{B})$ with $\tilde{A} = \sum(\mathrm{Re}\omega_j)\tilde{\lambda}_j$ and $\tilde{B} = \sum(\mathrm{Im}\omega_j)\tilde{\lambda}_j$] around $B = 0$ (respectively, $\tilde{B} = 0$). Terms containing odd B's and $\tilde{B}$'s are purely imaginary and they do not appear in inequality (6). Choose l so small that the higher-order terms are negligible.

Note that $R(\mathcal{F}) \subset \mathcal{F}$. In fact, the properties (i) and (iii) are obviously kept.[4,10] So I now discuss (ii). For simplicity I restrict myself to the recursion formula (3) of Migdal type in this Letter. Assume that $f^{(n)}(\sigma(z)v\tau(\omega)\tilde{v})$ is analytic in $D_{2^n l}$ and satisfies the bound (6) in this region. With use of the invariance of dv_1 and the property (i) of $\mathcal{F}$, I have

$$f^{(n+1)}(\sigma(z)v\tau(\omega)\tilde{v}) = \mathcal{N}^{-1}\left[\int f^{(n)}(\sigma(\tfrac{1}{2}z)v\tau(\tfrac{1}{2}\omega)v_1^{-1}) f^{(n)}(\sigma(\tfrac{1}{2}z)v_1\tau(\tfrac{1}{2}\omega)\tilde{v})\, dv_1\right]^2, \tag{8a}$$

$$\mathcal{N} = \left[\int f^{(n)}(v_1)^2\, dv_1\right]^2. \tag{8b}$$

This identity means that $f^{(n+1)}(\sigma(z)v\tau(\omega)\tilde{v})$ is analytic in $D_{2^{n+1}l}$ and satisfies the same bound (6) in this larger region. Note that $\{f^{(n)}(v)\}$ depend only on spec(v) since $\{f^{(n)}\}$ are class functions.

Theorem 1.—(1) Let $\mathrm{spec}(v) = \{\exp i\theta_1, \ldots, \exp i\theta_{N-1}, \exp - i\sum_1^{N-1}\theta_i\}$. Then for all $n \geq 0$,

$$f^{(n)}(v) \geq \exp\left[-(\beta N/2)\sum_1^{N-1}[\theta_i]_{2\pi}^2\right], \tag{9}$$

where $[\theta]_{2\pi} = \theta \bmod 2\pi$, $|[\theta]_{2\pi}| \leq \pi$. (2) Let $z_i = \phi_i \in R$ and $\omega_j = \theta_j \in R$ for all i and j. Then for all $n \geq n_0$,

$$(p!q!)^{-1}\left|\frac{\partial^{|p|+|q|}}{\partial\phi^p\partial\theta^q} f^{(n)}(\sigma v\tau\tilde{v})\right| \leq \mathrm{const}\times[\beta C(N)]^{(|p|+|q|)/2} \tag{10}$$

uniformly in v, $\tilde{v}$, ϕ, and θ, where $p = (p_1, p_2, \ldots, p_s)$ and $|p| = \sum p_i$, etc.

Proof: (1) Set $\sigma = 1$ and define $\tau(\omega) = \mathrm{diag}\{\exp(i\omega_1), \ldots, \exp(i\omega_{N-1}), \exp(-i\sum_1^{N-1}\omega_i)\}$, with $|\mathrm{Im}\omega_i| \leq \epsilon$. In this case $C(N) \leq N[1 + O(\epsilon^2)]$ in the bound (6) and this is inherited by all $f^{(n)}$. Assume $|\theta_i| \leq \pi$, and define an analytic function of one complex variable ζ by

$$g(\zeta) = f^{(n)}(\tau(\zeta\theta)\exp\left[(\beta N/2)(1+\epsilon)\left(\sum_1^{N-1}\theta_i^2\right)\zeta^2\right]$$

in the region $\Lambda_\epsilon = \{\zeta \in C; |\mathrm{Re}\zeta| \le 1, |\mathrm{Im}\zeta| \le \epsilon/\pi\}$. So $g(0)=1$ and

$$|g(\zeta)| \le g(\mathrm{Re}\zeta)\exp\{-(\beta N/2)[\epsilon - O(\epsilon^2)](\textstyle\sum \theta_i^2)(\mathrm{Im}\zeta)^2\}.$$

The maximal principle of analytic function means that $g(1)=g(-1) \ge g(0)=1$. Let $\epsilon \to 0$. (2) Choose n_0 so large that $2^{n_0}l \ge [C(N)\beta]^{-1/2}$. By the Cauchy integral formula, represent the left-hand side (LHS) of inequality (10) in terms of contour integrals along the contours $|z_i| = |\omega_j| = [C(N)\beta]^{-1/2}$ after setting $f^{(n)} = f^{(n)}(\sigma(\phi+z)v\tau(\theta+\omega)\tilde{v})$. So the bounds (5) and (6) complete the proof.[4,7] Q.E.D.

Now setting $\tau(\theta) = \mathrm{diag}\{\exp(i\theta), 1, \ldots, 1, \exp(-i\theta)\} \in SU(N)$, I consider

$$f^{(n+1)}(v\tau(\omega)) = \mathcal{N}^{-1}[\textstyle\int f^{(n)}(v\tau(\frac{1}{2}\omega)v_1^{-1}) f^{(n)}(v_1\tau(\frac{1}{2}\omega))\,dv_1]^2, \tag{11a}$$

$$\mathcal{N} + [\textstyle\int f^{(n)}(v_1)^2\,dv_1]^2. \tag{11b}$$

Define

$$\beta_v^{(n)}(a) = 2a^{-2}\ln\left|\frac{f^{(n)}(v\tau(ia))}{f^{(n)}(v)}\right| \tag{12}$$

which is real analytic in a ($|a| \le \epsilon$) and $v \in SU(N)$ for all $n \ge n_0$, and is even in a. The small positive constant ϵ does not depend on n ($\ge n_0$). These facts are easily proved by expanding $f^{(n)}(v\tau(ia))$ around $a=0$ and using Theorem 1. Note that

$$\beta_v^{(n)} = \beta_v^{(n)}(0) = -(\partial^2/\partial\theta^2)\ln f^{(n)}(v\tau(\theta))|_0, \tag{13}$$

and thus it is easily seen that

$$|\beta_v^{(n)}(a) - \beta_v^{(n)}| \le \mathrm{const}\times a^2, \tag{14a}$$

$$|\beta_v^{(n)} - \beta_{v'}^{(n)}| \le \mathrm{const}\times ||v-v'|| \tag{14b}$$

uniformly in $n \ge n_0$, $v, v' \in SU(N)$, and $a \in [-\epsilon, \epsilon]$. Setting $\omega = ia$, I take the absolute values of both sides of Eq. (11a), and use Eq. (12) and inequality (14a):

$$f^{(n+1)}(v)[1 + \tfrac{1}{2}a^2\beta_v^{(n+1)} + O(a^4)] \le (\mathcal{N})^{-1}[\textstyle\int f^{(n)}(vv_1^{-1}) f^{(n)}(v_1)[1 + \frac{1}{4}a^2[\beta^{(n)} - \Delta_n(v_1;v)] + O(a^4)]\,dv_1]^2, \tag{15}$$

where

$$\beta^{(n)} = \sup_v \beta_v^{(n)}, \quad \Delta_n(v_1;v) = \beta^{(n)} - \tfrac{1}{2}(\beta_{v_1^{-1}v}^{(n)} + \beta_{v_1}^{(n)}) \ge 0.$$

Choosing $v \in G$ such that $\beta^{(n+1)} = \beta_v^{(n+1)}$, I calculate $0 \le$ [RHS of inequality (15) − LHS of inequality (15)]$/a^2$ and let $a \to 0$. Thus I have

$$\beta^{(n+1)} \le \beta^{(n)} - \frac{\int f^{(n)}(vv_1^{-1}) f^{(n)}(v_1)\Delta_n(v_1;v)\,dv_1}{\int f^{(n)}(vv_1^{-1}) f^{(n)}(v_1)\,dv_1}, \tag{16}$$

which means that $\beta^{(n+1)} \le \beta^{(n)}$. Assume that $\lim\beta^{(n)} = \beta_c > 0$. $f^{(n)}(v\tau(\theta))$ is periodic in θ. Then if $f^{(n)} \ne 1$, there exist v_n and $\tilde{v}_n$ such that $\beta_{v_n}^{(n)} > 0$ and $\beta_{\tilde{v}_n}^{(n)} < 0$. By Theorem 1-(1), there exists a strictly positive K uniformly in n such that

$$\beta^{(n+1)} \le \beta^{(n)} - K\int \Delta_n(v_1;v)\,dv_1 = \beta^{(n)} - K\int[\beta^{(n)} - \beta_{v_1}^{(n)}]\,dv_1,$$

where $\beta^{(n)} \ge \beta_c > 0$. Inequality (14b) then implies that $\beta^{(n+1)} \le \beta^{(n)} - \kappa$ ($\kappa > 0$) uniformly in n, a contradiction. Then $\beta^{(n)} \to 0$. Choosing $n_0 = n_0(\beta)$ large, I can now assume that $(0 \le)\ 1 - f^{(n_0)}(v) \le \exp(-L)$, $L \gg 1$. By applying an asymptotic estimate to Eq. (3), it is found[4,10] that

$$0 \le 1 - f^{(n)}(v) \le \exp(-\tilde{L}2^{n-n_0}) \tag{17}$$

uniformly in n ($\ge n_0(\beta)$) and $v \in G$, with a positive constant $\tilde{L}$. Here 2^n is the size of area of the nth block plaquette Λ and one may regard $\partial\Lambda$ as a Wilson loop. One can easily estimate the coefficients of the character expansion of $f^{(n)}(v)$, with the help of inequality (17).

Theorem 2.—(1)

$$\lim f^{(n)} = 1. \tag{18}$$

(2) The string tension is strictly positive.

This method is extended to $G=U(N)$ and to the recursion formulas of Kadanoff type. A disappointing aspect of these approximate formulas is that the lattice $(QED)_4$ $[G=U(1)]$ confines fermions within these formalisms, as was already discussed in another place.[7] I feel that the present method of analysis can be applied to the real systems by knowing what is lost by these approximations. I hope that I can report on this problem in the near future (Ref. 8; see also Tomboulis[11]).

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