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Node-Avoiding Lévy Flight: A Numerical Test of the ϵ Expansion

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We study an extension of Lévy flight to include self-repulsion in the path of the walk. We call the extension node-avoiding Lévy flight and we show its equivalence to the $n \to 0$ limit of a statistical mechanical model for a magnetic system with long-range interactions between the spins. By use of this equivalence we are able to make a detailed comparison between the results of the ϵ expansion for the magnetic model, a Monte Carlo simulation of the Lévy flight model, and the results of a Flory-type argument. This is the first comparison of the ϵ expansion for $\epsilon << 1$ with a numerical simulation for any model. Some speculations are made on applications of the model of node-avoiding Lévy flight.

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Lévy flight is a random walk in which[1,2] the step length l is a random variable with (continuous) probability distribution proportional to $1/l^{1+\mu}$ (independent of spatial dimensionality d). In the interesting case in which $0 < \mu < 2$ the properties of such a random walk are strikingly different from those of ordinary random walks. For example,[2] the Hausdorff-Besicovitch dimension of the "trail" consisting of the end points of the steps is μ in the limit of a large number of steps (whereas it is 2 for ordinary random walks). Thus if one associates a mass with the end points of the steps, one has a mass distribution with dimension less than 2. It was this feature which motivated the introduction of the model,[1,3] which was originally intended to illustrate how a fractal mass distribution could account for clustering of matter in the universe.

In the present paper we consider a self-avoiding extension of the Lévy flight model. While we will suggest some possible experimental realizations of such models below, our emphasis here is on the use of one such model of self-avoiding Lévy flight to study the convergence of the ϵ expansion for $\epsilon << 1$ numerically for the first time.

We will consider random walks on a lattice which will be taken to be hypercubic for definiteness. We will use the same algorithm for generation of the walks which is used to define Lévy flight on a lattice[4,5] except for the self-avoiding feature: At each step, a direction for the next step is chosen at random and a step length l is chosen so that the ensemble of step lengths will approach the distribution

$$p(l) = \frac{a-1}{a} \sum_{n=0}^{\infty} a^{-n} \delta_{l,b^n}. \tag{1}$$

If the algorithm allows the resulting step, whether that step intersects some part of the path or not, then one has Lévy flight on a lattice as discussed in Refs. 5 and 6. As shown there, this Lévy flight has fractal dimension

$$D = \mu = \ln a / \ln b$$

if $\mu < 2$. Several self-avoiding constraints might be considered. Here we will study node-avoiding Lévy flight defined so that the step selected as described above is rejected if the position at the end of any proposed step intersects the position of the end of the previous step in the walk. The meaning of this constraint is illustrated in Fig. 1, where a disallowed step on a square lattice is shown. In each case, if a step is disallowed, the entire walk is discarded from the statistical sample, so that the resulting ensemble of walks weights the number of possible Lévy flights subject to the constraint only with the product of step-length distribution factors for each step taken from Eq. (1) above. Thus we are considering "equilibrium" ensembles in the sense that the self-avoiding walk describes an equilibrium ensemble while the "true" self-avoiding walk does not.[6,7] "Kinetic" versions of these models, analogous to the true self-avoiding walk, can be defined but we will not consider them here.

We can establish an equivalent magnetic model for node-avoiding Lévy flight: In qualitative field theoretic terms it is clear that the essential critical features of the problem will be reproduced by a model in which the propagator corresponding to the Gaussian model in magnetic systems is of the form $(q^\mu + r)^{-1}$ rather than $(q^2+r)^{-1}$ as it is for systems with short-range interactions. Here $r \propto T - T_c$, where T and T_c are respectively the temperature and critical temperature of the magnetic model. (One way to see this is to note that it is shown in Ref. 4 that the diffusion operator Dq^2 becomes Dq^μ for Lévy flight.) If, in a lattice model, we randomize among orthogonal directions on the lattice at each step, then it is not hard to show that if the diffusive propagator is to have the required form in momentum space, then the step-length distribution must be as in Eq. (1).

With this formulation, the development of a magnetic model yielding the statistics of node-avoiding Lévy flight proceeds as follows: Consider the model described by

$$-\beta H = \sum_{i,j} K_{ij} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{2}$$

Here β is the reciprocal of the temperature, H is the Hamiltonian, and the S's are n-component vector spins of length $\sqrt{n}$ on the lattice sites. K_{ij} is zero unless i and j lie along one of the d orthogonal directions in the lattice and is $K/r_{ij}^{1+\mu}$, where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the distance between sites i and j, when they lie along a coordinate axis. As in the Sarma[8,9] derivation of the equivalence of the nearest-neighbor n-vector model to a self-avoiding walk in the limit $n \to 0$, we consider the high-temperature expansion of the partition function for this model in the $n \to 0$ limit. As in Refs. 8 and 9,

$$Z = \prod_i \int d\Omega_i [\exp(-\beta H)]/\Omega^N$$

is 1 in the $n \to 0$ limit, and defining

$$\langle \ldots \rangle = \prod_i \int d\Omega_i [\exp(-\beta H)(\ldots)]/\Omega^N,$$

$$\langle \ldots \rangle_0 = \prod_i \int d\Omega_i (\ldots)/\Omega^N,$$

one has

$$\langle S_i^\alpha S_j^\alpha \rangle = \sum_N K^N \bar{\eta}_N(r_{ij}),$$

in which

$$\bar{\eta}_N(r_{ij}) = \sum_{\{\mathbf{r}_{lm}\}} \prod_{lm} p_{lm} \eta_N(r_{ij}, \{\mathbf{r}_{lm}\}).$$

Here the sum is over all combinations of step lengths given by $\{\mathbf{r}_{lm}\}$. The factors $p_{lm} = r_{lm}^{-(1+\mu)}$ provide the correct weighting for node-avoiding Lévy flight as explained above. $\eta_N(r_{ij}, \{\mathbf{r}_{lm}\})$ is the number of node-avoiding walks between $\mathbf{r}_i$ and $\mathbf{r}_j$ with N steps of lengths given by $\{\mathbf{r}_{lm}\}$. This relation establishes that the statistics of node-avoiding Lévy flight can be obtained from the statistical mechanical model described by Eq. (2) in exactly the same way that similar equivalences are established for the self-avoiding and biased walk problems.[8-10]

The ϵ expansion for the model described in Eq. (2)

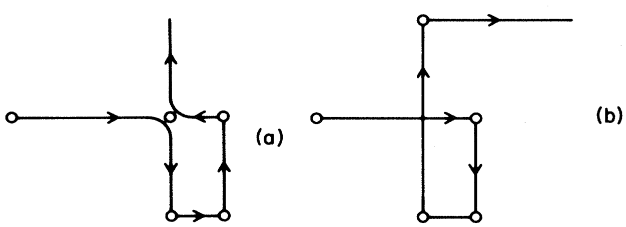

FIG. 1. Illustration of the constraint which defines node-avoiding Lévy flight. Node-avoiding flight excludes paths of the type illustrated in (a) but not those shown in (b).

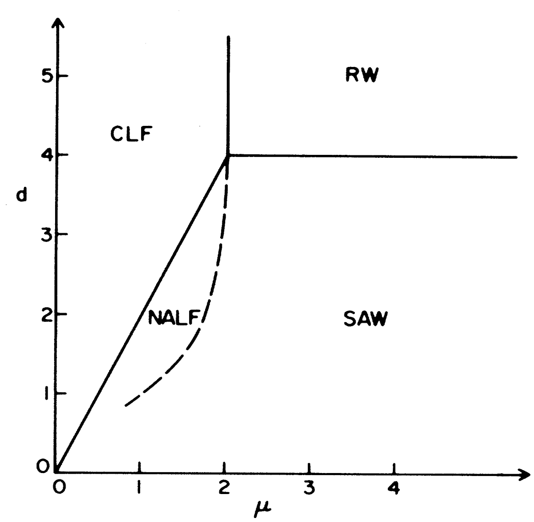

FIG. 2. d-μ diagram showing regions in which classical Lévy flight (CLF), random walk (RW), node-avoiding Lévy flight (NALF), and self-avoiding walk (SAW) behavior is expected.

has been studied by Fisher, Ma, and Nickel[11] and by Sak.[12, 13] A diagram showing the predicted behavior for various d and μ is given in Fig. 2. (Here we show only those aspects of the results of these references which can be expected to hold in the limit that the number n of spin components goes to zero.) We note four regions in the diagram corresponding to different critical behavior (associated with the asymptotic behavior of the walk in the limit of a large number of steps). The four regions are as follows.

Classical Lévy flight (CLF).—In this region ($\mu < 2$ and $d > 2\mu$) the self-avoiding constraint is predicted not to affect the results of Hughes, Schlesinger, and Montroll[4, 5] for the lattice version of the Lévy flight model. The diffusive propagator for this model takes the form $(q^{\mu}+r)^{-1}$ for the equivalent magnetic model consistent with Eq. (32) of Ref. 5. This form of the propagator is also consistent with the value $\eta = 2-\mu$ reported in Eq. (3) of Ref. 12 for the magnetic model. The value $\gamma = 1$ [Ref. 12, Eq. (3)] implies[9] that the number of classical Lévy flights of N steps approaches a constant to the power N as N becomes large. One must interpret the value $\nu = 1/\mu$ found[12] for this region with some care: The pair-correlation function for the magnetic model falls off algebraically (for all temperatures) consistent with the result $\langle R^2\rangle \to \infty$ reported in Refs. 4 and 5. The exponent ν determines the singularity in ξ, the correlation length scale needed to scale the correlation function which itself decays algebraically.

Random walk (RW).—Here ($\mu > 2$ and $d > 4$) one expects ordinary random walk behavior with $\nu = \frac{1}{2}$ and $\gamma = 1$.

Self-avoiding random walk (SAW).—In this region ($\mu > 2-\eta_{\mathrm{SAW}}$ and $d < 4$) the exponents should be those of the ordinary self-avoiding walk. Thus the coherence length exponent will be very near the Flory[9] value. For $d = 2$, the $n = 0$ values of the exponents are accurately known.[14–16]

Node-avoiding Lévy flight (NALF).—In the region $d/2 < \mu < 2-\eta_{\mathrm{SAW}}$ one expects the "long-range" exponents found in Ref. 11. In particular,

$$\frac{1}{\nu} = \left[1 - \frac{2\mu-d}{4\mu} - \frac{40}{512}\left(\frac{2\mu-d}{\mu}\right)^2 \frac{3-\mu^2}{4}\right]\mu + \dots . \tag{3}$$

The main interest here is in the ϵ expansion for the exponents in the NALF region, as displayed in Eq. (3). The ϵ expansion gives the critical exponents as a series in $\epsilon = 2\mu - d$ as one moves into the NALF region from the NALF-CLF boundary at $d = 2\mu$. It is the fact that the parameter μ is at our disposal in a simulation which provides the unique opportunity to use this model to compare the ϵ expansion for this model with Monte Carlo simulations with $\epsilon << 1$. (It

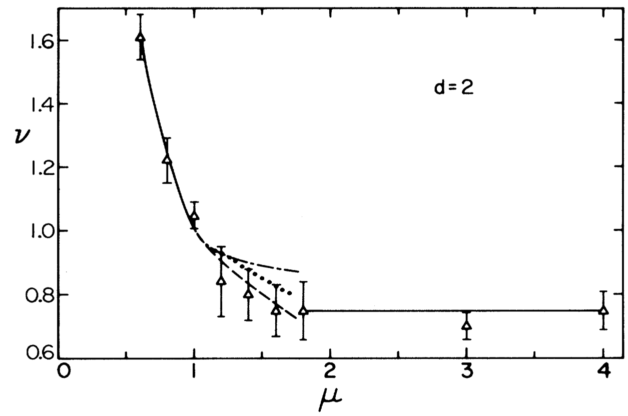

FIG. 3. Correlation length exponent ν vs μ for $d = 2$. The solid line for $\mu < 1$ is the classical Lévy flight prediction $\nu = 1/\mu$. The horizontal line for $\mu > 2-\eta_{\mathrm{SAW}} = 1.8$ is the Flory exponent ($=0.75$). The dashed line is the ϵ-expansion prediction to order ϵ. The dash-dotted line is the reciprocal of the first two terms of Eq. (3), while the dotted line is the expansion of the reciprocal of the first two terms of Eq. (3) to order ϵ^2. The triangles with error bars are our Monte Carlo results.

should be noted that the determination of the critical exponent ν from a Monte Carlo simulation of the walk is not entirely straightforward here because $\langle R^2\rangle$ diverges for these walks. We will provide details of our techniques elsewhere.) In Fig. 3 we show results for the exponent ν from Monte Carlo simulations of node-avoiding Lévy flight on the square lattice ($d = 2$) where they are compared with the known results for $\mu < 1$ and $\mu > 2-\eta_{\mathrm{SAW}}$ and with first- and second-order expansions in $\epsilon = 2\mu - d$ [Eq. (3)] in the region $1 < \mu < 2-\mu_{\mathrm{SAW}}$. In second order in ϵ we show both the reciprocal of the first two terms in Eq. (3) and also the reciprocal of Eq. (3) expanded to second order in ϵ in Fig. 3. We have also compared our Monte Carlo results with the "$\Delta\sigma$ expansion" of Ref. 11 and find that it agrees less well with the Monte Carlo results than the ϵ expansion results shown. The Monte Carlo calculations are for a minimum of 1000 walks of 50 steps for a given value of μ and, in most cases, much more (e.g., 13 000 walks for $\mu = 1$). The error bars are somewhat subjective because the occasional large steps make the run-to-run fluctuations rather large and unpredictable. They represent a conservative estimate of the errors if they are Gaussian distributed but do not take account of any possible systematic errors. We note that both the first-order ϵ expansion and the second-order expansion of the reciprocal of Eq. (3) agree with our simulation to within the estimated error bars over the whole range from $\mu = 1$ to $\mu = 2-\eta_{\mathrm{SAW}}$. On the other hand, if the reciprocal of Eq. (3) is used without expansion, then the second-order results appear to disagree with the simulations. Over most of the range of μ in both expansions, using only the first-order term in ϵ appears to give closer agreement with the simulations than does the second-order expansion. At the point $\mu = 2 - \eta_{\mathrm{SAW}}$ we find, using[16] $2 - \eta_{\mathrm{SAW}}$, that $\nu = 0.681$ to first order in ϵ, and that to second order in ϵ the reciprocal of Eq. (3) gives $\nu = 0.784$ so that at this point the second-order result is better than the first-order one if we expand the reciprocal of Eq. (3).

One can also produce an argument line that of Flory[9] for the exponent ν in this type of random walk: Using an argument closely similar to that described in Ref. 9 for $d < 2\mu$ gives $R \propto N^{3/(d+\mu)}$, where R is the radius of gyration of the coil so that $\nu = 3/(d+\mu)$, while $\nu = 1/\mu$ for $d > 2\mu$. These "Flory" values are in worse agreement with our Monte Carlo results than the ϵ-expansion values.

Finally, we offer a few speculative remarks on possible physical realizations of node-avoiding Lévy flight. One can imagine an "organism" which moves in a space of dimension d in which a fractal of dimension D is embedded. Initially, the mass of the fractal is to be regarded as "nutrient" for the organism. The organism moves in straight lines until it encounters part of the fractal. If that part of the fractal still contains nutrient, the organism consumes it and then starts to move in a straight line in another randomly selected direction. If the nutrient on the part of the fractal which the organism encounters is already consumed, then the organism dies. It then follows, if $D + 1 < d$, that the surviving organisms perform Lévy flight with $\mu = d - D - 1$. Similar scenarios might occur in the transport of chemically reactive species in a medium in which the reactant is thinly dispersed with dimension D.

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