Local Power Conservation for Linear Wave Propagation in an Inhomogeneous Plasma

B. D. McVey[(a)]
Plasma Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

and

R. S. Sund and J. E. Scharer
Electrical and Computer Engineering Department, University of Wisconsin, Madison, Wisconsin 53706
(Received 16 July 1984)

An expression for local power absorption for linear wave propagation in a nonuniform hot magnetoplasma is derived from fundamental principles. The power-absorption definition is used to obtain a local power-conservation relation for a one-dimensional configuration. The formalism is applied to wave propagation in the ion cyclotron range of frequencies where strong damping and mode-conversion processes are present.

PACS numbers: 52.35.Fp, 52.50.Gj

In this Letter we present a definition of local power absorption for linear wave propagation in a spatially dispersive, collisionless, magnetized plasma. The definition is generally valid within the context of a linear wave analysis and provides an extension beyond the limitations of weak damping or single-mode treatments. In the limit of weak damping, power-conservation theorems have been developed for a single wave propagating in a uniform plasma and used to treat weakly nonuniform plasmas.[1,2] More recently, conservation relations have been presented in conjunction with differential equations which model wave propagation and mode conversion near the second ion-cyclotron resonance.[3,4] The formalism we present encompasses the previous work and the precise definition of local power absorption provides a clear extension to other applications such as higher-harmonic or minority-ion heating.

We first present the definition of local power absorption based on the nonlocal wave-particle interaction. The definition is developed under the assumptions of the Vlasov theory of plasma waves.[5] Using this definition we outline the derivation of an expression for the local steady-state power absorption for time-harmonic waves in a plasma with one-dimensional nonuniformity. The result, presented in Eq. (9), is valid to second order in the ratio of gyroradius to wavelength and is therefore applicable to multimode propagation in the ion-cyclotron range of frequencies. Next, we indicate how such an expression may be used to derive a corresponding local conservation relation. Finally, we examine the limit of the homogeneous plasma. We prove that the fundamental definition of power absorption which we employ, Eq. (1), is correct in the homogeneous case. This fact, along with other evidence presented in this paper, indicates that our form is also valid in the inhomogeneous case.

We define the local power absorption $P(\mathbf{r})$ as the time-average rate of change of the energy of the group of particles which pass through $\mathbf{r}$. $P(\mathbf{r})$ must equal the average rate of work done by the wave field on these particles. Consider a set of plasma particles with coordinates $(\mathbf{r},\mathbf{v})$ at time t. Let $\tau=t-t'$ and let $\mathbf{r}'(\mathbf{r},\mathbf{v},\tau)$, and $\mathbf{v}'(\mathbf{r},\mathbf{v},\tau)$ define the unperturbed trajectory these particles travel; thus $\mathbf{r}'=\mathbf{r}$ and $\mathbf{v}'=\mathbf{v}$ at $\tau=0$. The instantaneous rate of work that the field does on such a particle at time t' is given by $q\mathbf{E}(\mathbf{r}',t')\cdot\mathbf{v}'$. To obtain the average rate of work performed on all the particles in the set, the single particle is weighted by the distribution function evaluated at $\mathbf{r}',\mathbf{v}',t'$ and the time average of the product is computed as t' varies. Finally, the total power absorbed by all the particles passing through $\mathbf{r}$ is obtained by an integration over $\mathbf{v}$ as follows:

$$P(\mathbf{r})=\int d^3v\,\langle q\mathbf{E}(\mathbf{r}',t')\cdot\mathbf{v}'f_1^*(\mathbf{r}',\mathbf{v}',t')\rangle, \tag{1}$$

where angular brackets denote the time average over t' and the asterisk indicates the complex conjugate. The complex power is defined in Eq. (1) and the real power transfer between the wave fields and the plasma particles is determined by $\frac{1}{2}\mathrm{Re}\{P(\mathbf{r})\}$. Since the equilibrium distribution function is stationary in time, only the perturbed distribution function appears in Eq. (1). It should be noted that $\mathbf{E}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})$ cannot represent the local power absorption in a hot plasma, since the work done by $\mathbf{E}$ is not evaluated along the trajectories of the plasma particles.

The perturbed distribution function appearing in Eq. (1) may be computed in the usual manner by solution of the linearized Vlasov equation by the method of characteristics,

$$f_1(\mathbf{r}',\mathbf{v}',t')=-(q/m)\int_{-\infty}^{t'}dt''[\mathbf{E}(\mathbf{r}'',t'')+\mathbf{v}''\times\mathbf{B}(\mathbf{r}'',t'')]\cdot\partial f_0/\partial\mathbf{v}''. \tag{2}$$

The characteristics are identical to the unperturbed orbits, $\mathbf{r}''=\mathbf{r}'(\mathbf{r},\mathbf{v},\tau'=t-t'')$, etc.

Consider now a one-dimensional configuration in which the plasma is uniform in the y and z directions and is immersed in a z-directed magnetic field $\mathbf{B}_0$. The magnitude of $\mathbf{B}_0$, the density, and the temperature are functions of x. We assume that the scale lengths of these quantities are large compared to the transverse scale length of the wave field. The wave electric field is assumed to be of the form $\mathbf{E}(\mathbf{r},t)=\mathbf{E}(x)\times\exp(ik_z z-i\omega t)$ where $\mathbf{E}(x)$ is complex and k_z and ω are taken to be real. To first order in ϵ ($\epsilon=B_0'/B_0$) the unperturbed particle orbitals for such a situation are as follows[5]:

$$v_x'=g+g^*,\quad v_y'=-i(g-g^*)\;2\epsilon v_+v_-/\omega_c,$$
$$v_z'=v_z, \tag{3}$$

where

$$g=v_+e^{i\omega_0\tau}+i(\epsilon/\omega_0)[-v_xv_+e^{i\omega_0\tau}+v_+^2e^{i2\omega_0\tau}],$$
$$v_\pm=(v_x\pm iv_y)/2,\quad \omega_0=\omega_c+\epsilon v_y,$$

and

$$\Delta x=x'-x=-\int_0^\tau v_x'(t')\,dt',\quad z'=z-v_z\tau. \tag{4}$$

The equilibrium distribution is assumed to have the form

$$f_0(x_g,\mathbf{v})=f_\mathrm{M}(x,v^2)+(v_y/\omega_c)\partial f_\mathrm{M}(x,v^2)/\partial x$$

where $f_\mathrm{M}(x,v^2)$ is a local isotropic Maxwellian. Since the wave fields may be nonsinusoidal in the x direction, a Taylor series is used to represent the electric field along particle orbits:

$$\mathbf{E}(x')=\mathbf{E}(x)+\Delta x\mathbf{E}'(x)+\dots. \tag{5}$$

The remainder of the derivation is outlined as follows: The perturbed distribution function is evaluated via Eq. (2) with use of the equilibrium distribution f_0, the wave field of Eq. (5), and the trajectories of Eqs. (3) and (4). Collecting terms by powers of $\exp(i\omega_0\tau)$ we obtain

$$f_1(\mathbf{r}',\mathbf{v}',t')$$
$$=i(q/m)e^{i(k_zz'-\omega t')}\sum_m e^{im\omega_0\tau}b_m/\sigma_m, \tag{6}$$

where $\sigma_m=\omega+m\omega_0-k_zv_z$ and the coefficients b_m are functions of $\mathbf{E}$, $\mathbf{E}'$, . . . , $\mathbf{v}$, f_M, f_M', and ϵ. Similarly,

$$\mathbf{E}(\mathbf{r}',t')\cdot\mathbf{v}'$$
$$=e^{i(k_zz'-\omega t')}\sum_n e^{in\omega_0\tau}a_n(\mathbf{E},\mathbf{E}',\dots,\mathbf{v}), \tag{7}$$

where the coefficients a_n are comprised of those terms in b_n which have $2f_\mathrm{M}/u^2$ as a multiplying factor.

Using these results we form the product

$$q\mathbf{E}\cdot\mathbf{v}'f_1^*=-(iq/m)\sum_{n,m}e^{i(n-m)\omega_0\tau}a_nb_m^*/\sigma_m^*. \tag{8}$$

To calculate P as defined by Eq. (1) we take the time average of Eq. (8) which annihilates all but the $m=n$ terms. An integration over velocity space completes the calculation. Using Eq. (5) truncated to second order yields $P(x)=i\omega\epsilon_0\sum_s\sum_n p_n(x)$ where

$$p_0=2\Delta\{-\xi_0Z_0^{*\prime}[2|E_z|^2+\rho^2(|E_z'|^2+E_zE_z^{*\prime\prime}+\mathrm{c.c.})]+2\rho^2Z_0^*|E_y'|^2+\rho Z_0^{*\prime}(E_zE_y^{*\prime}+\mathrm{c.c.})$$
$$-\epsilon\rho[Z_0^{*\prime}(E_yE_z^*+\mathrm{c.c.})+2\rho Z_0^*(|E_y|^2)']\}-(d/dx)[\Delta\rho^2\xi_0Z_0^{*\prime}](|E_z|^2)', \tag{9a}$$

$$p_1=\Delta(\tfrac{1}{2}Z_1^*\{4|E_+|^2+\rho^2[2(|E_+|^2)''+E_+(E_x^*-3iE_y^*)''+\mathrm{c.c.}]\}-i\rho Z_1^{*\prime}(E_+E_z^{*\prime}-\mathrm{c.c.})$$
$$-p^2\xi_1Z_1^{*\prime}|E_z'|^2+\epsilon p^2Z_1^*/2(E_+E_-^{*\prime}+\mathrm{c.c.}))+2(d/dx)[\Delta\rho^2Z_1^*](|E_+|^2)', \tag{9b}$$

$$p_2=\Delta\rho^2Z_2^*[|E_+'|^2+\epsilon(|E_+|^2)'], \tag{9c}$$

where $\Delta=\omega_p^2/(4\omega k_zu)$, Z_n is the plasma dispersion function of argument $\xi_n=(\omega-n\omega_c)/k_zu$, $Z_n'=dZ_n/d\xi_n$, ρ is the thermal gyroradius, and c.c. is the complex conjugate of the previous term. The corresponding p_{-n}'s are obtained from the p_n's by the replacements $n\to -n$, $E_+\to E_-$, $i\to -i$, $\epsilon/k_z\to -\epsilon/k_z$. Several physical processes can be identified in $P(x)$. The first three terms in p_0 are respectively a Landau damping term, a transit-time magnetic pumping term, and a cross term. The lead terms in p_1 and p_2 are the fundamental and second-harmonic cyclotron damping terms. The expression is valid to first order in gradients of the magnetic field, density, and temperature, and to second order in the scale length of the wave field, all of which are long compared to an ion gyroradius.

A power-conservation relation appropriate for the slab configuration may be derived from the expression for the local power absorption. The following familiar quadratic form is readily derived from the time-harmonic form of Maxwell's equations:

$$(d/dx)(E_yH_z^*-E_zH_y^*)+i\omega\epsilon_0|\mathbf{E}|^2-i\omega\mu_0|\mathbf{H}|^2+\mathbf{E}\cdot\mathbf{J}^*=0. \tag{10}$$

Next we observe that $\mathbf{E}\cdot\mathbf{J}^*$ may be extracted from P as follows. Repeated application of the differential identity $A'B=(AB)'-AB'$ to any expression, F, written in a form such as Eq. (9) yields an equivalent expression of the form $F=\mathbf{E}\cdot\mathbf{G}^*-dH/dx$ where G contains derivatives of $\mathbf{E}$. For example, the term $\Delta\rho^2Z_2^*|E_+'|^2$ appearing in Eq. (9c) becomes $\mathbf{E}\cdot[-(\hat{\mathbf{x}}-i\hat{\mathbf{y}})(\Delta\rho^2Z_2E_+')']^*+(d/dx)(\Delta\rho^2Z_2^*E_+E_+^{*\prime})$. The first contributes to $\mathbf{E}\cdot\mathbf{J}^*$ and the second to dS/dx. A proper definition of P must satisfy the condition that the above procedure yields

$$P(x)=\mathbf{E}(x)\cdot\mathbf{J}^*(x)-dS(x)/dx, \qquad (11)$$

where $\mathbf{J}^*=\int d^3v\,\mathbf{v}f_1^*$. That is, the factor multiplying $\mathbf{E}$ obtained in this manner must be identical to the conjugate current density obtained from the first velocity moment of f_1. If this is true then a power-conservation relation for wave propagation in a slab geometry is obtained by use of Eq. (11) to replace $\mathbf{E}\cdot\mathbf{J}^*$ in Eq. (10). $S(x)$ is interpreted as the kinetic flux due to coherent motion of the plasma particles in the wave field.

We have verified that the P given by Eq. (9) does in fact satisfy Eq. (11). Length constraints prevent us from listing in this Letter the expressions for $\mathbf{J}$ and S associated with Eq. (9). In the context of the slab model, with the further approximation $E_z=0$, our results for P, S, and $\mathbf{J}$ encompass those obtained by other researchers.[3,4]

We now consider a single wave mode propagating in a uniform, thermally isotropic, Maxwellian plasma. In such a situation the x variation of the wave field is given by $\mathbf{E}_0\exp(ik_\perp x)$ where $k_\perp$ is in general a complex solution of the hot-plasma dispersion relation for real frequency ω. After transformation to cylindrical velocity-space coordinates $(v_\perp,\phi,v_z)$, the perturbed distribution functions take the form

$$f_1(\mathbf{r}',\mathbf{v}',t')=Ce^{i(\zeta+\gamma\sin\phi)}\sum_n e^{-in(\omega_c\tau+\phi)}\alpha_n/\sigma_{-n}, \qquad (12)$$

where

$$\alpha_n=(v_\perp/2)[E_-J_{n+1}(\gamma)+E_+J_{n-1}(\gamma)]+v_zE_zJ_n(\gamma),\gamma=k_\perp v_\perp/\omega_c,$$

$E_\pm=E_x\pm iE_y$, $C=2iqf_\mathrm{M}(v)/(mu^2)$, $\zeta=k_\perp x+k_zz'-\omega t'$, and u = thermal velocity. The scalar product $\mathbf{E}(\mathbf{r}',t')\cdot\mathbf{v}'$ has the same form as f_1 with the replacements $C\rightarrow 1,\sigma_n\rightarrow 1$. Notice that we no longer require a small gyroradius. Using these results we form the product

$$q\mathbf{E}\cdot\mathbf{v}'f_1^*=-(2iq^2/mu^2)f_\mathrm{M}e^{-y}\sum_{n,m}e^{i(m-n)(\omega_c\tau+\phi)}\alpha_n\alpha_m^*/\sigma_{-m}^*, \qquad (13)$$

where $y=2k_{\perp i}(x+v_\perp\sin\phi/\omega_c)$. To calculate P as defined in Eq. (1) we take the time average of Eq. (13) which annihilates all but the $m=n$ terms. Integrating over velocity space gives

$$P=-\sum_s(2iq^2/mu^2)\int d^3v\,f_\mathrm{M}e^{-y}\sum_n|\alpha_n|^2/\sigma_{-n}^*, \qquad (14)$$

where s designates particle species. The pole defined by $v_z=(\omega^*-n\omega_c)/k_z$ will be slightly below the real v_z axis if ω is given a small positive imaginary part. For purely real ω the v_z integration contour is deformed to pass above the pole; thus Eq. (14) reveals $\mathrm{Re}P(x)$ to be a positive definite quantity. In fact, the positive definite nature of $\mathrm{Re}\{P\}$ in the homogeneous limit does not rely on the single-mode representation since then $b_n=2a_nf_m/u^2$ in Eqs. (6)–(8). This is another important property which a correct formulation of local power absorption must satisfy.

It is worthwhile to compare Eq. (14) with the corresponding result of weak-damping theory. If the expression in Eq. (13) is evaluated at $\tau=0$ and an integration over velocity space is performed, we obtain $\mathbf{E}\cdot\mathbf{J}^*$. In this case the ϕ integral selects $m=n$. The result becomes identical to Eq. (14) when both are evaluated at $k_\perp=k_{\perp r}$, and thus

$$\lim_{k_{\perp i}\rightarrow 0}\tfrac{1}{2}\mathrm{Re}P(x)=\tfrac{1}{2}\omega E_i^*\epsilon_{ij}^{a}(k_{\perp r})E_j, \qquad (15)$$

where $\epsilon_{ij}^{(a)\mathrm{H}}$ is the (anti-)Hermitian part of the equivalent well-known dielectric tensor.[5] The right-hand side is recognized as the usual weak-damping result.

A direct consequence of Eq. (15) is that the kinetic flux associated with Eq. (1) must also agree with the results of conventional weak-damping theory in the corresponding limits. From Eq. (14) we observe that the x dependence of P goes like $\exp(-2k_{\perp i}x)$, and thus $k_{\perp i}P=-\frac{1}{2}dP/dx$. Applying this fact to an expansion of P about $k_{\perp r}$, $P=P(k_{\perp r})+ik_{\perp i}\partial P/\partial k_{\perp r}$, and using Eqs. (11) and (15), we can show that

$$\lim_{k_{\perp i}\rightarrow 0}\tfrac{1}{2}\mathrm{Re}S(x)=-\frac{\omega}{4}E_i^*\frac{\partial\epsilon_{ij}^{h}(k_{\perp r})}{\partial k_{\perp r}}E_j. \qquad (16)$$

Thus we see that the expression we have presented for local power absorption is motivated by the fundamental definition of the work performed on a charged particle in an electric field. It is always positive definite in a homogeneous plasma and the corresponding conservation relation reduces to the correct weak-damping limit. Furthermore, the explicit form derived for a particular inhomogeneous plasma is seen to obey a proper conservation relation. While a general proof of the correctness of our formulation remains to be found, the above properties lend strong support to its validity.

The research by one of us (B.D.M.) was supported by the Department of Energy under Contract No. DE-AC02-78ET51013 and that by two of us (R.S.) and (J.E.S.) was supported by the National Science Foundation under Grant No. ECS-8212173.

[(a)]Present address: Los Alamos National Laboratory, MS E531, Los Alamos, New Mex. 87545.

[1]A. Bers, in *Plasma Physics,* edited by C. DeWitt and J. Peyraud, Summer School of Theoretical Physics, Les Houches, 1972 (Gordon and Breach, New York, 1975).

[2]V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasmas* (Addison-Wesley, New York, 1964), Appendix B (temporal dispersion), Appendix C (spatial dispersion).

[3]P. L. Colestock and R. J. Kashuba, Nucl. Fusion **23**, 763 (1983).

[4]S. C. Chiu and T. K. Mau, Nucl. Fusion **23**, 1613 (1983).

[5]N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973), Chap. 8.