Cyclotron Motion in a Microwave Cavity: Possible Shifts of the Measured Electron g Factor

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The interaction of a bound electron with the radiation field produced by the image charges that represent a surrounding cavity produces a shift in its orbital frequency and in its radiative decay time. We calculate the frequency shift and the change in the damping constant for a cyclotron motion at the midpoint of a lossy, cylindrical cavity. The frequency shift can easily be so large as to have important consequences for the University of Washington $g-2$ measurements.

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The effect of apparatus upon precision measurements is a topic of much recent interest.[1-5] A paradigm is the University of Washington $g-2$ experiment[6] with a single electron, where the Penning-trap electrodes form a crude microwave cavity which slightly shifts the cyclotron frequency and modifies the cyclotron decay rate. Such a change in the cyclotron damping constant has been observed experimentally.[7,8] The g factor is proportional to the ratio of the free-space values of the spin and cyclotron frequencies. Thus a cavity shift Δ of either frequency gives a systematic error $\Delta g/g=\Delta/\omega_c$. The present experimental precision[8] is upset if the cavity shift is larger than $\Delta/\omega_c \approx 5\times 10^{-12}$. Such a shift would not be made evident by comparison with the theoretical value[9] since its uncertainty corresponds to $\Delta/\omega_c \approx 140\times 10^{-12}$. Initial claims[2,3] that there are large cavity-induced shifts of the spin frequency led to a demonstration[4] that the exact apparatus of quantum electrodynamics reduces to the classical theory with no change in the spin frequency to within a high order of accuracy (10^{-20}). The g factor is nonetheless affected by the cavity, but by shifts in the cyclotron frequency rather than in the spin frequency. We summarize here our exact classical results[10] for the changes in the cyclotron motion at the midpoint of a lossy, cylindrical cavity whose symmetry axis is parallel to the magnetic field.[11] This calculable model should give a useful indication of the size of the effects in the hyperbolic electrodes used in the $g-2$ experiments. We find that the decay constant can be substantially altered and, more importantly, that the shift in the cyclotron frequency $\Delta\omega_c$ can be much larger than previously estimated,[4,5] seriously impacting the measured g factor. As a byproduct, we describe the shifts in the cyclotron motion at the midplane between two infinite, parallel plates. Our work, in conjunction with Ref. 4, can also be generalized to treat cavity shifts on atomic systems now being studied.[1]

With neglect of insignificant image magnetic forces, the presence of a surrounding metallic cavity alters the equation of motion to read

$$\dot{\mathbf{v}}-\boldsymbol{\omega}_c\times\mathbf{v}+(e/m)\nabla V(\mathbf{r})+\tfrac{1}{2}\gamma_c\mathbf{v}=(e/m)\mathbf{E}'(\mathbf{r}). \quad (1)$$

Here $\mathbf{E}'(\mathbf{r})$ is the electric field at the position $\mathbf{r}(t)$ of the electron which is produced by the effective image charges that represent the cavity walls. It is the electric field acting on the particle omitting the trap field $[-\nabla V(\mathbf{r})]$ and also excluding the self-field of the particle itself. This self-field is accounted for by use of the observed (free-space) electron mass m, and by employment of the free-space damping constant $\gamma_c(\omega_c)=4e\omega_c^2/3mc^3$. It is convenient to split the field $\mathbf{E}'$ into transverse and longitudinal parts, $\mathbf{E}'={}^{(L)}\mathbf{E}'+{}^{(T)}\mathbf{E}'$. The longitudinal piece ${}^{(L)}\mathbf{E}'$ produces an insignificant harmonic force that may be neglected.[10]

The transverse electric field may be expressed as

$$ {}^{(T)}E_k'(t,\mathbf{r}) = -\frac{\partial}{\partial t}\int dt' \sum_{l=1}^{3} D_{kl}'(t-t';\mathbf{r},\mathbf{r}(t'))\,e v_l(t')/c^2. \quad (2)$$

Here $D_{kl}'(t-t';\mathbf{r},\mathbf{r}')$ is the retarded, transverse, radiation-gauge Green's-function alteration brought about by the trap electrodes. Adding it to the free-space Green's function produces the full Green's function which obeys the relevant boundary conditions on the cavity walls. Since the electron is confined to a small region near the center of the trap, it suffices to set $\mathbf{r}=0=\mathbf{r}(t')$ in Eq. (2). If we adopt complex coordinates and take the Fourier transform according to $v(t)=v_x(t)-iv_y(t)\sim e^{-i\omega t}$, the equation of motion (1) yields the condition

$$\omega-\omega_c'+i\gamma_c/2=-\omega r_0\tilde{D}_{xx}'(\omega;0,0), \quad (3)$$

where $r_0=e^2/mc^2$ is the classical electron radius, and $\tilde{D}_{xx}'(\omega;0,0)$ is the Fourier transform of the Green's-function alteration in Eq. (2). The effect of the trapping potential is to replace the cyclotron frequency ω_c by the modified frequency[12] ω_c' on the left-hand side of Eq. (3).

Let us first ignore the renormalization problem so that the Green's-function correction $\tilde{D}_{xx}'(\omega;0,0)$ in Eq. (3) is replaced by the full Green's function, and the decay constant γ_c is omitted. In this case we may express the Green's function by a mode sum to obtain

$$\omega - \omega_c' = \omega \sum_N \frac{\lambda_N^2}{\omega^2 + i\omega\Gamma_N - \omega_N^2}. \tag{4}$$

Here ω_N is the eigenfrequency of the Nth mode and Γ_N is the decay constant of this mode, with $Q_N = \omega_N/\Gamma_N$ the corresponding quality factor. Formula (4) expresses the frequency shift of the cyclotron motion, which is essentially harmonic, in terms of its interaction with the infinite number of cavity modes of the radiation field, each of whose amplitudes is a harmonic oscillator. A simple dimensional argument shows that λ_N^2 is of order $r_0/d^3c^2 \sim (r_0/d)\omega_N^2$, where d is the characteristic size of the cavity. Therefore, away from any cavity resonance $\omega = \omega_N$, there is a small frequency shift of the order of $(\omega - \omega_c')/\omega \sim r_0/d$. However, near a cavity resonance, the frequency shift (from this one mode) can be as large as $(\omega - \omega_c')/\omega \sim \pm(r_0/d)(\omega_N/\Gamma_N) = \pm(r_0/d)Q_N$, which is much larger. This frequency shift disappears exactly on resonance, but then there is a large change in the cyclotron decay constant of order $(r_0/d)Q_N\omega \sim \gamma_c Q_N$.

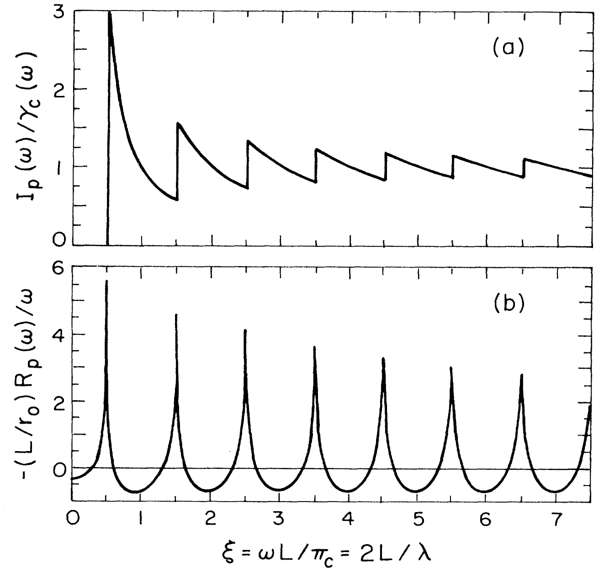

FIG. 1. (a) The decay constant for a charged particle moving in the midplane between two perfect conducting planes. (b) Frequency shift for $\delta/L = 2\times 10^{-3}$.

In the limit of an infinitely large cavity, the imaginary part of the right-hand side of Eq. (4) must reproduce the free-space decay constant $-i\gamma_c/2$. But in this limit, the real part of the right-hand side of Eq. (4) is infinite since it contains the reactive effect of the self-field of the electron. Hence the formal mode sum in Eq. (4) must be renormalized by subtracting out the real part of the free-space limit. Since this is a delicate operation, we use instead the previous formula (3), which expresses the (complex) frequency shift in terms of the alteration $\tilde{D}_{xx}'(\omega;0,0)$ of the Green's function brought about by the presence of the cavity.

We take approximate account of dissipation by replacing the individual cavity widths Γ_N with an average value Γ. Referring to the mode sum (4), we see that since $\Gamma^2 << \omega_N^2$, this is tantamount to replacing the frequency ω by the complex number $\omega + \frac{1}{2}i\Gamma$. To determine unambiguously the renormalized alteration $\tilde{D}_{xx}'(\omega;0,0)$, we note that the limit in which the cavity radius R is taken to infinity yields a geometry with two parallel, infinite conducting planes a distance $2L$ apart. Thus we express the Green's function as the sum of the Green's function for the parallel-plate problem plus the solution to the homogeneous wave equation which corrects for the presence of the cylindrical wall. This gives

$$\omega - \omega_c' = -\tfrac{1}{2}iI(\omega) + R(\omega) = -\tfrac{1}{2}i\gamma_c(\omega) + \omega[\Sigma_P(\omega + \tfrac{1}{2}i\Gamma) + \Sigma_S(\omega + \tfrac{1}{2}i\Gamma)], \tag{5}$$

where Σ_P is the parallel-plate contribution to Eq. (3), and Σ_S is the correction due to the cylindrical side of the cavity. The imaginary part $I(\omega)$ is the cavity-modified cyclotron decay rate at frequency ω, and the real part $R(\omega)$ the cavity shift of the cyclotron frequency. Since these changes are very small, ω can be replaced by ω_c' on the right-hand side of Eq. (5). Since the Green's function for the two–parallel-plate geometry can be expressed as an infinite sum of image contributions, the removal of the self-field term is now trivial: One simply omits the direct contribution from the sum.

Using the method of images we obtain

$$\Sigma_P(\omega) = \frac{r_0}{L}\ln(1 + e^{2i\omega L/c}) - \frac{r_0}{L}\sum_{n=1}^{\infty}(-1)^n\left\{e^{2in\omega L/c}\left(\frac{ic}{2n^2L\omega} - \frac{c^2}{4n^3L^2\omega^2}\right) + \frac{c^2}{4n^3L^2\omega^2}\right\}. \tag{6}$$

The cavity dissipation in this parallel-plate case can be modeled by writing $\Gamma = \omega\delta/L$, where δ is the skin depth of the conducting plates. It is convenient to describe the frequency by the dimensionless variable $\xi = \omega L/\pi c = 2L/\lambda$, which is the (fractional) number of wavelengths that fit between the plates. The decay constant $I_P(\omega)$ for perfectly conducting plates ($\Gamma = 0$) given by Eq. (6) is plotted in Fig. 1(a). With $\xi < \frac{1}{2}$, less than half a wavelength fits between the plates. In this case, electromagnetic waves cannot propagate between the plates, the electron cannot radiate, and $I_P(\omega) = 0$. The decay constant $I_P(\omega)$ jumps discontinuously as ξ passes through thresholds for propagating waves at odd half-integers. As ξ becomes large there is no obstacle to radiation, and $I_P(\omega)$ approaches the free-space value γ_c. The effect of nonvanishing dissipation is to smooth the sharp discontinuities and to produce a small contribution below the first threshold $\xi=\frac{1}{2}$. In Fig. 1(b) we plot $R_P(\omega)$, taking $\delta/L=2\times10^{-3}$. This frequency shift vanishes as $\omega\rightarrow\infty$. The large peaks appear when ξ is an odd half-integer because here the retardation phase exactly cancels the alternating signs of the image charges, and the resultant infinite-image sum would be the divergent sum of $1/n$ if it were not for the damping resulting from cavity dissipation, which produces instead a large logarithm.

The alteration of the Green's function brought about the presence of the circular side of radius R can be expressed in terms of an infinite sum over the axial standing waves which fit between the two end-cap planes. The wave numbers of the waves which do not vanish at the midplane location of the electron are given by $k_n=(n+\frac{1}{2})\pi/L$, where $n=0,1,2,\ldots$. With ω below the first axial threshold, $\xi<\frac{1}{2}$, the radial waves are exponentially damped with the damping constant $\mu_n=(k_n^2-\omega^2/c^2)^{1/2}$. In terms of this decomposition, the cavity-side addition to the complex frequency shift (5) is given by

$$\Sigma_S(\omega)=-\frac{r_0}{L}\sum_{n=0}^{\infty}\left\{\frac{K_1'(\mu_nR)}{I_1'(\mu_nR)}+\frac{k_n^2c^2}{\omega^2}\left[\frac{K_1(\mu_nR)}{I_1(\mu_nR)}-\frac{K_1(k_nR)}{I_1(k_nR)}\right]\right\}, \tag{7}$$

where the prime denotes a derivative. The first ratio of Bessel functions is the TE contribution; the terms in the large parentheses are the TM contribution. When ω is near the nth threshold, μ_n becomes small, and the nth term in the sum (7) has a large logarithmic contribution that cancels the large logarithm in the parallel-plate term. As ω passes the threshold, μ_n becomes a negative imaginary number. In the limit of vanishing dissipation ($\Gamma=0$), the imaginary part of the Bessel function ratios cancels the imaginary part of the parallel-plate term. Past a threshold the Bessel functions in the denominator can vanish, producing poles corresponding to the normal modes of the cavity. The replacement $\omega\rightarrow\omega+\frac{1}{2}i\Gamma$ changes these poles into Lorentzian forms of width Γ. The sum in Eq. (7) converges very rapidly: For large n, $\mu_n\sim k_n\sim n\pi/L$, and it is exponentially damped. Thus the sum is easily calculated on a digital computer. Adding the result to the previous parallel-plate contribution gives the complete shift of Eq. (5).

Experiments are generally performed in the region $3.5<\xi<4.5$, which we examine in detail for a cylindrical cavity with $R/L=1.5$ to model the presently employed hyperbolic traps that have ring/end-cap distance ratio of $\sqrt{2}$. Our results are shown in Fig. 2, using $Q=1000$. We see that the damping constant $I(\omega)$ varies from $0.06\gamma_c(\omega)$ to $21\gamma_c(\omega)$. A Q of about 1000 is required to make possible the decrease in the damping constant by the factor 10 which has been observed.[8] To set the scale for the frequency shifts, we note that in the traps $r_0/L\approx8\times10^{-13}$, while the current experimental precision is equivalent to a shift in the cyclotron frequency given by $\Delta\omega_c/\omega_c\approx5\times10^{-12}$. We see from the figure that, on this scale, very large shifts occur in the vicinity of the normal-mode frequencies, shifts as large as $\Delta\omega_c/\omega_c\approx90\times10^{-12}$. For the most part the shifts are on the order of $\Delta\omega_c/\omega_c\approx8\times10^{-12}$ in the regions between the resonances where the damping constant is small. In view of the uncertainties in the theoretical value of the anomaly, a shift as large as $\Delta\omega_c/\omega_c=140\times10^{-12}$ would not be revealed by comparison of the experimental and theoretical results for the anomaly. We conclude that an experimental search for this systematic effect should be made to confirm the present value of the g factor of the electron.

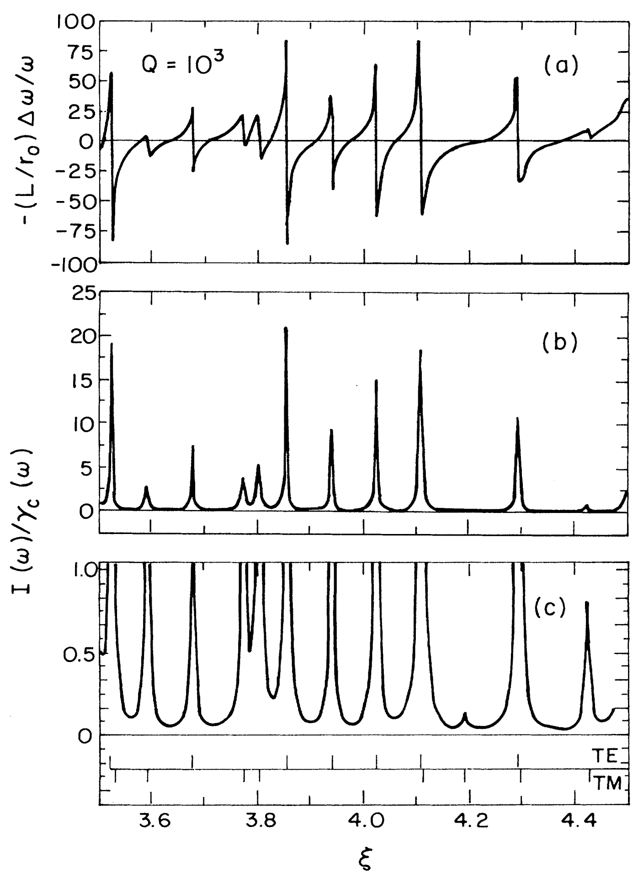

FIG. 2. Cavity effects for $R/L=1.5$ and $Q=1000$. The ticks denote the positions of the TE and TM modes. (a) Frequency shift. (b) Decay constant. (c) Magnified section of the smaller values of the decay constant.

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