QCD at Large N_c—Skyrme or the Bag?

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A framework for extracting the low-energy dynamics of pseudoscalar mesons from QCD at large N is developed and, to leading order in the decoupling of heavy mesons, the pure pseudoscalar theory is calculated truncated to four derivatives. The soliton is found not to be manifestly stabilized by the four-derivative terms. Under a plausible assumption about the classical solution for the scalar meson field, a natural realization of the previously proposed topological soliton bag model is obtained.

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There is little doubt presently that QCD is indeed a fundamental theory of hadrons. Yet, as far as low-energy hadron dynamics is concerned, the evidence for this is mainly qualitative and comes largely from considerations of QCD in the limit of a large number of colors.[1,2] At the same time, it is well known that the low-energy interactions of hadrons are very successfully described by phenomenological chiral Lagrangians. One of them, the old Skyrme model,[3] has been recently reviewed[4,5] from a modern point of view. It was shown that a topological soliton, known to appear in this model, has indeed the qualitative properties of the baryon[5] as originally conjectured, but quantitative agreement is not quite satisfactory.[6] The appearance of a solitonic baryon fits very nicely into a previously conjectured picture of the large-N limit.[2] Altogether, it seems very plausible that the large-N limit is appropriate for establishment of a connection between QCD and the low-energy physics. Some more phenomenological attempts in this direction can be found.[7]

In this Letter I will show, under a few technical, essentially large-N assumptions, that to leading order in the decoupling of heavy mesons, pure pseudoscalar low-energy dynamics is completely calculable directly from QCD, and it will be calculated truncated to four derivatives. Among the calculated four-derivative terms, apart from the usually assumed Skyrme term, there are two other terms of opposite sign, making the contribution of the quartic term to the static energy not manifestly positive. Yet, under an assumption about the classical solution for the scalar meson field, the result leads to a natural realization of the previously proposed picture of a topological soliton bag model[8] with massless quarks and gluons within an unbroken vacuum bubble in the center of a topological soliton. Further, I derive an expression for the effective potential, an invariant function of quark bilinears whose minimization would lead in principle to determination of the ground state. While the expression for the potential is not explicit enough to enable one to demonstrate spontaneous breakdown of the chiral symmetry without further assumptions, assuming that the chiral symmetry is indeed broken,[9] we obtain an interesting expression for the order parameter $\langle \bar{q}q \rangle$, linking it to the chiral-symmetry–breaking scale and the constituent quark mass.

Consider QCD in the limit of a large number of colors. We will assume that the theory confines at arbitrary large N. In the absence of the explicit quark-mass term the theory has global $U(N_F) \otimes U(N_F)$ which, below some scale Λ, is spontaneously broken down to diagonal $U(N_F)$. Consider the following quark bilinear:

$$\phi^i_j(x) = \bar{q}^i_R(x) q_{Lj}(x).$$

The ground state of the theory below the chiral-symmetry–breaking scale will be characterized by operator ϕ^i_j being frozen to its large expectation value. The lowest-energy states above the ground state are Goldstone bosons of broken chiral symmetry—pseudoscalar mesons. Fluctuations of $\phi^i_j(x)$ around its expectation value $\langle\sigma\rangle$ have correct quantum numbers to be identified as scalar and pseudoscalar mesons. It is natural, therefore, to define what is meant by scalar and pseudoscalar mesons by enforcing the following identification:

$$\mathbf{\Phi}(x) = V^\dagger(x)\sigma(x)V^\dagger. \tag{1}$$

$\sigma(x)$ is the Hermitian matrix of scalar mesons while the pseudoscalar field matrix is $U = V \times V = \exp i(2/F_\pi)\mathbf{\Pi}(c)$ and transforms according to a nonlinear realization of $SU(N_F) \otimes SU(N_F)$ in a standard way. The partition function for QCD with the measure appropriately extended to include integration over collective fields defined above is

$$\int [dU\, d\sigma][dG_\mu] d\bar{q}\, dq\, \delta(\bar{q}_L q_R - V\sigma V)\delta(\bar{q}_R q_L - V^\dagger\sigma V^\dagger)\exp\left(iS_{\rm QCD} - im\int \bar{q}q\right). \tag{2}$$

The current-quark mass term is included above. In principle, one could now imagine separating the measure into a short- and long-wavelength part with respect to some physical scale Λ (the chiral-symmetry–breaking scale). Above this scale one has a weakly coupled theory of quarks and gluons moving with high relative momenta. There are no bound states. A good description of the physics in this region is given by perturbation in the color coupling. Below the scale Λ the color forces are becoming rapidly strong, the chiral symmetry breaks down, and the bound states are formed. At large N the bound states are mesons, interact weakly, and are certainly much better candidates for the physical states than strongly interacting quarks and gluons. Clearly, quantizing around the perturbative QCD vacuum is not appropriate anymore—QCD as usually written should be rewritten such that quantization of the small vibrations around the right ground state is made possible. The first step in this direction was to rewrite the action as in (2); the second is to integrate out gluons. The resulting action $G[J]$ [10] is a functional of a color current and has, therefore, a full local chiral invariance. It contains all kinds of higher-dimension composite operators as well as the effective potential which, following Ref. 9, is assumed to be some chiral-invariant function of quark bilinears $\phi_j^i(x)$ defined above. If we now expand composite operators around their vacuum values and quantize the small vibrations around the vacuum values, we get an effective theory of mesons interacting with massive quarks. The question is whether we can somehow extract the low-energy dynamics of the lightest mesons, pseudoscalars, without really explicitly calculating the effective action due to gluons, $G[J]$. The observation which makes this indeed possible is that the only piece of the QCD action not having a full local chiral invariance is the quark kinetic term and the measure. This means that by making a particular chiral redefinition of the quark fields one can extract the pseudoscalar mode from the quark fields and rotate the Goldstone bosons away from $G[J]$. Let us, therefore, make this particular chiral change of basis, passing from chiral quarks to massive ones:

$$q_L = V^\dagger(x)\, q_L^U, \quad q_R = V(x)\, q_R^U. \tag{3}$$

In this gauge the pseudoscalars are localized to couple only through the quark kinetic term and, because they are introduced through a finite chiral rotation of the quarks, would also couple to the quark measure [$\ln J(U)$ term in (4) below]. $G[J]$ now contains only operators capable of creating the heavy mesons from the vacuum. Those, however, are not excited in the limit of very low energies and being interested in the lowest-energy excitations above the ground state we keep those operators frozen to their vacuum values. This amounts to keeping only the zero-momentum term, i.e., potential, and we will drop out a complicated and unknown piece of $G[J]$ containing chiral invariants made out of the heavy currents and derivatives of the scalar meson-type quark bilinears. From the viewpoint of calculating the effective, pure pseudoscalar theory this approximation means that we neglect contributions to the coefficients of this theory due to exchanges of the heavy mesons. Those contributions are suppressed by inverse powers of the heavy-meson masses and, as long as the momenta involved are much smaller then the exchanged mass, are not important. If we integrate the quarks out, the long-wavelength part of the partition function becomes

$$\int [dU\, d\sigma]\, dS\, dP \exp\Big\{N_c \operatorname{Tr}\ln[\mathscr{D}(U) - (mU + S + i\gamma^5 P)] + \ln J[U] + i\int [2\operatorname{tr}\mathbf{S}\cdot\boldsymbol{\sigma} - V_{\text{gluon}}(\sigma)]\Big\}, \tag{4}$$

where $\mathscr{D}(U) = i\gamma^\mu(\partial_\mu + V_\mu + \gamma^5 A_\mu)$, $V_\mu = \frac{1}{2}(V^\dagger \partial_\mu V + V\partial_\mu V^\dagger)$, and $A_\mu = \frac{1}{2}(V^\dagger\partial_\mu V - V\partial_\mu V^\dagger)$. The pair (S,P) of auxiliary fields is used to exponentiate the constraints.

Consider now the integral over the auxiliary fields P and S. At large N it is dominated by a stationary phase. Using the equations of the motion we find

$$P_{\text{cl}} = 0, \quad S_{\text{cl}} \equiv \Sigma(\sigma) = \partial V_{\text{gluon}}/\partial\sigma + \dots . \tag{5}$$

We will therefore approximately evaluate the integral over P and S, at large N, by the replacement $P = P_{\text{cl}}$ and $S = S_{\text{cl}}$. Within this approximation the partition function describing the low-energy dynamics of scalar and pseudoscalar mesons is obtained and reads

$$Z \approx \int [dU\, d\sigma] \exp iW_{\text{eff}}(U,\sigma),$$

where

$$iW_{\text{eff}}(U,\sigma) = N_c \operatorname{Tr}\ln\{\mathscr{D}(U) - [mU + \Sigma(\sigma)]\} + \ln J[U] + i\int [2\operatorname{tr}\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}(\sigma) - V_{\text{gluon}}(\sigma)]. \tag{6}$$

With $\Sigma(\sigma)$ defined in (5), this expression is our first result. It makes manifest where the pseudoscalars are localized and shows that pure pseudoscalar low-energy effective theory is, to leading order in the decoupling of heavy mesons, completely calculable given $\langle\Sigma\rangle$, i.e., the dynamical quark mass.

The fermion determinant written above is, of course, a formal object and needs to be defined through some regularization method. I use the proper-time method. It is easy to show that this regularization respects the vector gauge invariance.[11] The proper-time integration has to be truncated at some maximal momentum Λ, which is interpreted to be the chiral-symmetry–breaking scale. For the effective potential we find

$$V_{\text{eff}}(\sigma) \simeq \frac{N_c \Lambda^4}{32\pi^2} \operatorname{Tr} \int_1^\infty \frac{dS}{S^3} \exp[-(S/\Lambda^2)(mU+\Sigma)^\dagger (mU+\Sigma)] - 2\operatorname{tr}\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}(\sigma) + V_{\text{gluon}}(\sigma). \tag{7}$$

Minimization of the potential leads to the following interesting relation:

$$-\langle \bar{q}q \rangle \simeq \frac{N_c \Lambda^3}{4\pi^2} \left(\frac{m_Q}{\Lambda}\right) \int_1^\infty \frac{dS}{S^2} \exp\left[-S\left(\frac{m_Q}{\Lambda}\right)^2\right], \tag{8}$$

where $m_Q = m + \langle \partial V_{\text{gluon}}/\partial\sigma \rangle_{\sigma\neq 0}$ is the constituent-quark mass and $\langle \bar{q}q \rangle = N_F^{-1} \langle \bar{q}_R^E q_L^E \rangle$. It is not difficult to see that the nonvanishing current-quark mass leads necessarily to $\langle \bar{q}q \rangle \neq 0$ above ($\langle \bar{q}q \rangle = 0$ is not the stationary point in this case), but without knowing $V_{\text{gluon}}(\sigma)$ we are unable to prove whether symmetry breaking persists in the limit of zero bare mass for the quarks.

In order to calculate the fermion determinants in (6) and arrive at the promised low-energy theory, one has to calculate the corresponding heat kernel. While the quark measure we started with is defined with respect to the massless Dirac operator, it is the dynamical-symmetry breakdown which leads to the appearance of the kinetic term and nontopological four-derivative term for pseudoscalars, as our calculation will illustrate. If one imagines artifically turning off a dynamical-symmetry breakdown, only the Wess-Zumino phase, being of a topological origin, will survive. In the presence of some external vector and axial-vector fields, the anomalies will be correctly reproduced.[5,12] We calculate the heat kernel truncated to four derivatives by expanding it in powers of Σ/Λ around the massless kernel[13] and then calculating the contributions to four-derivative action to all orders in powers of Σ/Λ. As the final result of a straightforward but very lengthy calculation we obtain

$$\begin{aligned} W_{\text{eff}}(U,\Sigma) \simeq \int d^4x \Bigg\{ & \frac{F_\pi^2}{4} \operatorname{tr} \partial^\mu U^\dagger \partial_\mu U - i\frac{N_c}{48\pi^2} \int_0^1 d^5x\, \epsilon^{\mu\nu\rho\sigma} \operatorname{tr}(U^\dagger \partial_5 U U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U) \\ & + \frac{N_c}{192\pi^2}[1-\mathscr{K}(\Sigma^2)] \operatorname{tr}\{2\partial^2 U^\dagger \partial^2 U + \tfrac{1}{2}[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 - (\partial^\mu U^\dagger \partial_\mu U)^2\} \\ & + \frac{N_c \Lambda^3}{4\pi^2}\left(\frac{\Sigma}{\Lambda}\right) \int_1^\infty \frac{dS}{S^2} \exp\left[-S\left(\frac{\Sigma}{\Lambda}\right)^2\right] \operatorname{tr}\left[\frac{m}{2}(U^\dagger + U)\right] + O\left(\frac{m^2}{\Lambda^2}\right) + \ldots \Bigg\}, \end{aligned} \tag{9}$$

where $\mathscr{K}(\Sigma^2) \simeq 1$ for $\Sigma^2 \to 0$, while $\mathscr{K}(\Sigma^2) \simeq 0$ for $\langle \Sigma \rangle \simeq m_Q$. To leading order in the decoupling of heavy scalar mesons, the resulting pure pseudoscalar theory is given by the terms explicitly displayed above. From the requirement that the kinetic term for the pseudoscalars is properly normalized as above, we obtain the following condition:

$$\frac{N_c \Lambda^2}{16\pi^2} \left\{ \exp\left[-\left(\frac{m_Q}{\Lambda}\right)^2\right] + \sqrt{\pi}\frac{m_Q}{\Lambda} \operatorname{erf}\left(\frac{m_Q}{\Lambda}\right) - 1 + \int_{1/\Lambda^2}^{\sim 1/m_Q^2} \frac{dS}{2S^{3/2}} \sqrt{\pi} \frac{m_Q}{\Lambda^2} \operatorname{erf}(m_Q\sqrt{s}) \right\} \approx \frac{F_\pi^2}{4}. \tag{10}$$

In obtaining the last term in the brackets above, we have assumed that a certain proper-time integral is dominated by the region $s \lesssim m_Q^{-2}$. Then the kernel in the integrand, which is indeed exponentially suppressed for $s \gtrsim m_Q^{-2}$, can be well approximated by its small-s expansion. The details of calculation leading to expressions (8), (9), and (10) are too technical and I hope to present them elsewhere. The second term in (9) is the celebrated Wess-Zumino term[5,12] with $U = U(x,\tau)$ interpolating between $U = \exp[i(2/F_\pi)\Pi(x)]$ at $\tau = 1$ and $U = 1$ at $\tau = 0$. The overall factor multiplying the four-derivative terms in (9) as well as (10) above clearly indicates the chiral-symmetry breakdown, i.e. $\langle \Sigma \rangle \neq 0$, as the origin of the appearance of pseudoscalars. Among the four-derivative terms, the second (Skyrme term) and third contribute to the stability of the soliton. However, the first is of opposite sign and tends to destabilize the soliton.[14] It does not seem possible to claim that the quartic term will be manifestly positive and we conclude, therefore, that radiative corrections are not a likely mechanism for stabilization of the soliton. If, however, we assume that the classical solution for $\sigma(x)$, i.e. $\Sigma(\sigma)$, will interpolate between the unbroken vacuum at short distances and the normal vacuum $\langle \Sigma \rangle \neq 0$ at large distances, then the overall suppression factor multiplying the kinetic (10) and four-derivative terms acts as a natural space cutoff, essentially a step function, making the contribution of pseudoscalars rapidly vanish below a certain critical radius, and leaving a bubble of unbroken vacuum with massless quarks and gluons inside.[8] The topological soliton will in this case be stable against shrinking to zero size. The baryon number is the sum of the fraction carried by the quarks inside the bag and the topological charge carried by the chiral soliton, and is unity.[15]

Through our derivation we have seen the appearance of the two relations, (8) and (10), relating the four mass scales m_Q, Λ, $\langle \bar{q}q \rangle$, and F_π. If we take as an input $F_\pi = 95$ MeV and $m_\pi = 130$ MeV ($m = 8$ MeV) we calculate $\Lambda \simeq 667$ MeV and $m_Q \simeq 200$ MeV. The value of the quark mass comes out somewhat low. The realistic value is obtained by lowering the cutoff Λ to about 400 MeV. In this case the pion mass comes out off by factor of 2 unless the current-quark–mass input is taken well above ($m \simeq 30$ MeV) the currently favored value.[16] Indeed it might be reasonable within our framework not to expect that the mass of the pion is realistically accounted for.

A more detailed description of this work will be presented elsewhere.

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[10] $G(J)$ is defined as

$$\exp(iG[J]) = \int [dG_\mu] \exp i \int d^4x [-\tfrac{1}{4} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} - J_\mu^2 G^{2\mu}]$$

where $J_\mu^2 = g_c \bar{q} \gamma_\mu Q^2 q$ is the color current.

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[13] The general method for evaluating the massless kernel is explained in Ref. 10. One starts with the asymptotic expansion

$$\langle x | \exp[-\epsilon \mathscr{D}^2(U)] | x \rangle \simeq (16\pi^2\epsilon^2)^{-1}(1 + \epsilon h_1 + \epsilon^2 h_2 + \ldots);$$

then using the heat equation we get the following relation:

$$h_l(x,x) = l^{-1}(\mathscr{D}^2 h_{l-1})_{x=y} = [l(l+1)]^{-1}(\mathscr{D}^4 h_{l-2})_{x=y} + [l(l+1)]^{-1}[\gamma^\mu\gamma^\nu V_{\mu\nu} + 2\gamma^5 D \cdot A + 4A \cdot A] h_{l-1}(x,x)$$

which makes it possible to evaluate the coefficients, h_1 and h_2.

[14] One would like to know to what extend our result for the four-derivative terms is regularization dependent. It is therefore good to know that in the case $\langle \Sigma \rangle = \infty$ our result agrees with one obtained by dimensional regularization in a somewhat different content; see Eric D'Hoker and Edward Farhi, Massachusetts Institute of Technology Report No. CTP-1166, 1984 (unpublished).

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