Faddeev Calculation of Three-Nucleon Force Contribution to Triton Binding Energy

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The configuration-space Faddeev equations are solved for Hamiltonians that include the Tucson-Melbourne and Brazil three-nucleon forces. Convergence in terms of the number of three-body partial waves is established. First-order perturbation-theory results are shown to be inadequate. Both three-nucleon forces produce approximately 1.5 MeV additional binding, which overbinds the triton.

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Trinucleon (^{3}H and ^{3}He) bound-state calculations, using a model in which nonrelativistic nucleons interact via pairwise "realistic" potentials that reproduce nucleon-nucleon (*NN*) scattering data up to 300 MeV and properties of the deuteron, indicate that such a Hamiltonian underbinds the triton by 0.8–1.1 MeV and produces too large a charge radius by 0.1–0.2 fm.[1] Because of these and other discrepancies between model results and experiment, various groups have made estimates of the contribution of long-range three-nucleon forces to the binding of the triton.[2–7] These studies have dealt primarily with the Tucson-Melbourne (TM)[8] and Brazil (BR)[9] two-pion–exchange models of the three-body force, models which respect chiral constraints. (These are the models we have investigated, although a third approach in terms of an isobar constituent model employing a three-body isobar force of the Fujita-Miyazawa type[10] has been explored by Hajduk and Sauer.[11]) The diverse approximations used in estimating these three-nucleon force contributions to the triton binding energy[2–7] have given the appearance of significant discrepancies among some of the published calculations.[1,7] However, the results of Refs. 6 and 7 appear to have established the numerical accuracy of the first-order perturbation-theory estimates. Unfortunately, they imply a strong model dependence. If this is the case, then the assumption that we need consider only the long-range part of the three-body force is invalid. Thus, we are motivated to attack the full problem.

We have now solved the bound-state, configuration-space Faddeev equations to obtain numerically exact results for the ^{3}H binding energy using the TM and BR three-nucleon force models for a single value of the pion-nucleon form-factor cutoff Λ defined below. We find several novel results: (1) The TM model requires a nonperturbative treatment to obtain even qualitatively reliable numbers; (2) both three-body force models require that a large number of three-body channels (viz., 34, all two-body potential components with $J \leqslant 4$) be included in the calculation to ensure that the answer has converged; and (3) both three-body force models lead to overbinding of ^{3}H whether the underlying two-nucleon Hamiltonian is based upon the stiff Reid-soft-core (RSC)[12] or the softer Argonne (V14)[13] nucleon-nucleon (*NN*) potential model. Furthermore, we find no strong model dependence in the converged calculation, which supports the long-range two-pion–exchange assumption. We elaborate on these points below.

Without attempting to pass judgment on the appeal of the underlying philosophies of the competing methods for generating three-nucleon forces,[14] we shall adopt for the purposes of these theoretical investigations the TM (model independent) and BR (chiral Lagrangian) models as our *Ansätze*. Most emphasis in three-nucleon–force studies has been on the two-pion–exchange component of the potential, because the strong repulsion in the dominant *NN* interaction is expected to suppress the effectiveness of shorter-range components of the three-body force. These two models of the long-range, two-pion–exchange three-nucleon force have, in fact, the same functional form but differ significantly in the value of one parameter (c), which governs the size of the singular (δ function) part of the potential. For an analytic expression, we refer the reader to Refs. 8 and 9. However, we specify the model parameters which we have used in Table I. Note that the BR model with $c = 0.0$ is much less singular. In addition, we have used a pion mass of $\mu = 139.6$ MeV, a nuclear mass of $m = 6.726\mu$, a coupling constant of $g^2 = 179.7$, and a form-factor cutoff parameter of $\Lambda = 5.8\mu$. We emphasize that our purpose is not to argue the validity of either model but to produce benchmark trinucleon bound-state calculations, which can be used to determine the best means of exploring three-body force effects.

In order to evaluate the three-body force results which we have obtained, we first recall in Table II the binding energies and charge radii of the RSC and V14

TABLE I. Two-pion-exchange three-nucleon force parameters for the TM and BR models.

	μa	$\mu^3 b$	$\mu^3 c$	$\mu^3 d$
TM	1.130	−2.580	1.00	−0.7530
BR	1.048	−2.287	0.0	−0.7656

nucleon-nucleon potential models for 5, 9, 18, and 34 channels.[15] Note that the softer V14 potential calculation converges faster than that for the stiff RCS potential as a function of the number of the three-body channels. (The value of $J_{\max}$ and the parity of the *NN*-force partial waves for 5, 9, 18, and 34 channels, respectively, are $J \leqslant 1$, +; $J \leqslant 2$, +; $J \leqslant 2$, ±; and $J \leqslant 4$, ±.) The radii decrease as expected as the three-body binding energy increases.[1] (For a comparison with results of other groups, see Ref. 15.)

We have used three different decompositions of the Schrödinger equation into Faddeev equations in an effort to determine which method converges fastest when a three-body force is included in the Hamiltonian. (Lack of space prevents us from discussing in detail the numerical methods we use to solve the equations; see, for example, Payne *et al.*[16]) However, schematically we separate the Schrödinger equation [with Jacobi coordinates $\mathbf{x}_i = \mathbf{r}_j - \mathbf{r}_k$, $\mathbf{y}_i = (\mathbf{r}_j + \mathbf{r}_k)/2 - \mathbf{r}_i$],

$$H\Psi = E\Psi, \tag{1}$$

$$H = T + \sum_i V(x_i) + W(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{2}$$

$$\Psi = \psi(\mathbf{x}_1, \mathbf{y}_1) + \psi(\mathbf{x}_2, \mathbf{y}_2) + \psi(\mathbf{x}_3, \mathbf{y}_3), \tag{3}$$

$$= \psi_1 + \psi_2 + \psi_3, \tag{4}$$

as follows:

$$(T + V_i - E)\psi_i = -V_i(\psi_j + \psi_k) - W_i\Psi, \tag{5}$$

$$(T + V_i - E)\psi_i = -V_i(\psi_j + \psi_k) - W\psi_i, \tag{6}$$

$$(T + V_i - E)\psi_i = -V_i(\psi_j + \psi_k) - \tfrac{1}{3}W\Psi, \tag{7}$$

where $W = W_1 + W_2 + W_3$ as in Refs. 8 and 9. In each of these decompositions, the three-body force is retained on the right-hand side of the equations, so that the *NN* tensor force couples at most two channels on the left-hand side. We emphasize that each method of decomposing the three-body force in the Faddeev equations must lead to the same binding energy when we include all (or enough) partial waves. That is, each of Eqs. (5)–(7) is equivalent to the original Schrödinger equation.

To check our eigenvalue solutions we use the wave functions that we generate to calculate variational bounds $\langle H \rangle = \langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle$. By projecting $(V + W)$ in the same manner (same number of partial waves) that we employ in the Faddeev calculations, we can test our accuracy by means of

$$\langle \Psi | P \rangle \langle P | T + V + W | P \rangle \langle P | \Psi \rangle = \langle H_P \rangle .$$

TABLE II. Two-nucleon force results for the RSC and V14 potential models as a function of the number of channels.

		$-E_F$ (MeV)	$\langle r^2 \rangle_{\rm ch}^{1/2}(^3\mathrm{He})$ (fm)	$\langle r^2 \rangle_{\rm ch}^{1/2}(^3\mathrm{H})$ (fm)
RSC	5	7.02	1.89	1.70
	9	7.21	1.87	1.68
	18	7.23	1.87	1.68
	34	7.35	1.85	1.67
V14	5	7.44	1.86	1.68
	9	7.57	1.84	1.67
	18	7.57	1.84	1.67
	34	7.67	1.83	1.67
Expt.		8.48	1.69(3)	1.51(4)

When we do not project the three-nucleon force, we obtain a variational bound on the full (projected two-body plus full three-body) Hamiltonian. We list in Table III results for the three procedures [W_1, W, and $W/3$ corresponding to Eq. (5), Eq. (6), and Eq. (7)] as a function of the number of three-body channels (partial waves) for the V14-TM model comprised of the V14 *NN* potential plus the TM three-nucleon potential. (Clearly, $E_F = \langle H_P \rangle$ to an excellent approximation.) First, the results are incomplete for the W prescription because for eighteen channels (the first opportunity to include negative-parity *NN*-potential partial waves) the eigenvalue was several hundred megaelectronvolts. If enough three-body channels had been included, the same answer would have been obtained. But this anomalous result for eighteen channels caused us to reject this procedure for TM force calculations. (Results for the less singular BR potential were acceptable.) Second, although the Faddeev eigenvalue in the $W/3$ procedure tracks the full Hamiltonian expectation $\langle H \rangle$ better than that for W_1 in the V14-TM model, the opposite holds for the RSC-BR case (not shown). Thus, neither decomposition holds a clear advantage when fewer than 34 channels are included in the calculation. Third, both the W_1 and $W/3$ procedures result in the same $\langle H \rangle$ for 34 channels. The $W/3$ eigenvalue is closer (by 30 keV) for the TM force; for the BR force the eigenvalues are identical for 34 channels (but not for 18 channels).

Let us turn to the question of the validity of first-order perturbation-theory to estimate the size of the three-body force contribution to the triton binding energy. For the five-channel V14-TM model, Wiringa *et al.*[6] obtained a first-order perturbation-theory estimate for $\langle W \rangle$ of $E_1 = -0.14$ MeV ($\langle H \rangle = -7.42$

TABLE III. Triton binding energies (MeV) from the V14-TM model for the three Faddeev decompositions of Eqs. (5)–(7) as a function of the number of three-body channels.

	No. of channels	$-E_F$	$-\langle H_P \rangle$	$-\langle W_P \rangle$	$-\langle W \rangle$	$-\langle H \rangle$
W_1	5	8.26	8.26	1.98	1.93	8.22
	9	8.96	8.96	2.77	2.22	8.40
	18	9.49	9.50	3.29	2.88	9.09
	34	9.36	9.36	2.88	2.84	9.32
W	5	8.12			1.40	8.31
	9	8.64			1.61	8.49
	18	...			...	...
	34	...			...	...
$W/3$	5	7.82	7.83	0.69	1.23	8.36
	9	8.35	8.35	1.23	1.44	8.57
	18	9.11	9.12	2.61	2.61	9.11
	34	9.33	9.33	2.84	2.85	9.32

MeV) compared to our actual values for $\langle W \rangle$ with the W_1 procedure of -1.93 MeV (-8.22 MeV) and for $W/3$ of -1.23 MeV (-8.36 MeV). Ishikawa *et al.* agreed with Wiringa *et al.* for the five-channel RSC-TM model. In addition, they have made the first eighteen-channel estimate, obtaining $E_1 = -0.89$ MeV; this is to be compared to our $\langle W \rangle = -2.3$ MeV from the W_1 procedure. First-order perturbation theory appears completely inadequate to treat the TM force. For the RSC-BR model, Wiringa *et al.* found a value of $E_1 = -1.10$ MeV for the five-channel wave-function first-order perturbation estimate compared to our (five-channel) complete-solution values of $\langle W \rangle = -1.43$ MeV, -1.35 MeV, and -1.32 MeV for the W_1, W, and $W/3$ calculational procedures. First-order perturbation theory yields qualitatively correct results for the less singular BR three-body force.

To make this point clearer, we examine Hajduk's perturbation series[15] for $E = \langle \Psi | H | \Psi \rangle$:

$$\langle \Psi | H | \Psi \rangle = E_0 + E_1 + E_2 + E_3. \tag{8}$$

Here, we have $E_0 = \langle \Psi_0 | H_2 | \Psi_0 \rangle$, where H_2 is the NN-force Hamiltonian and Ψ_0 is its eigenfunction, and $E_1 = \langle \Psi_0 | W | \Psi_0 \rangle$, where $W = H - H_2$ is the three-body potential. If one *assumes* that $E_i = 0$ for $i \geqslant 4$, then $E_2 = 3(E - E_0) - 2E_1 - \overline{\Delta E}$ and $E_3 = -2(E - E_0) + E_1 + \overline{\Delta E}$, where $\overline{\Delta E} = \langle \Psi | W | \Psi \rangle$. Obviously this

TABLE IV. Perturbation series energies (MeV) as a function of the number of three-body channels for the V14-TM and RSC-BR models with the W_1 Faddeev decomposition.

No. of channels	5	9	18	34
	V14-TM			
$-E$	8.26	8.96	9.50	9.36
$-E_0$	7.44	7.57	7.57	7.67
$-E_1$	−0.13	0.35	0.92	0.76
$-E_2$	0.76	0.71	0.66	0.63
$-E_3$	0.20	0.34	0.35	0.29
	RSC-BR			
$-E$	7.64	8.77	8.69	8.89
$-E_0$	7.02	7.21	7.23	7.36
$-E_1$	0.49	1.13	1.10	1.17
$-E_2$	0.09	0.30	0.27	0.31
$-E_3$	0.05	0.13	0.08	0.06

TABLE V. The 5-channel, 9-channel, 18-channel, and 34-channel triton binding energies (negative of the W_1 eigenvalues) for four model combinations.

Model \ No. of channels	5	9	18	34
V14-TM	8.26	8.96	9.49	9.36
V14-BR	8.32	9.27	9.06	9.22
RSC-TM	7.55	8.33	8.93	8.86
RSC-BR	7.66	8.77	8.70	8.89

series can be easily generated only because we have the complete solution Ψ. However, comparing E_1, E_2, and E_3 permits one to understand just how well first-order perturbation theory works. In Table IV we list results for the V14-TM model as a function of the number of three-body channels for the W_1 procedure of Eq. (5). Clearly, there is no convergence; we do not find $E_3 << E_2 << E_1$. Furthermore, E_1 as a function of number of channels varies significantly, as was suggested by the work of Ref. 7. Also shown are results for the RSC-BR model. Here the first-order E_1 result is the dominant part of the $E - E_0$ difference. (Note also that nine channels give a reasonable approximation to the 34-channel result, because of the less singular nature of the BR three-body force.) Before leaving the perturbation-theory question, we point out that the strong model dependence of the five-channel result and the small value of E_1 for that approximation accounts for most of the disparate results referred to in the introduction. In particular, the results of Wiringa *et al.* and Ishikawa *et al.* are completely consistent with Table IV.

Finally, let us examine eigenvalues from the W_1 procedure, Eq. (5), for the four model combinations in Table V. First, note that the 0.4 (0.3) MeV difference between the 34-channel V14-TM (V14-BR) and RSC-TM (RSC-BR) eigenvalues is the same difference that is seen in Table II for the *NN*-force Hamiltonian eigenvalues. Both three-nucleon force models yield approximately 1.5 MeV additional binding. Second, the RSC-TM and RSC-BR 34-channel eigenvalues are very similar; the strong short-range repulsion of the RSC *NN* force does effectively suppress the contribution of the singular short-range behavior of the TM force (the c term). The softer V14 *NN* force allows differences in the two three-body force models to show themselves slightly. Third, we reiterate that one needs a full 34-channel calculation to obtain quantitatively reliable results. That is, the small (odd-wave) components of Ψ are needed to obtain an accurate result because of the strong odd-wave coupling of the three-body force. In this limit, one finds support in the lack of model dependence for the conjecture that the long-range, two-pion–exchange three-body force is the component of primary importance because of the strong short-range repulsion of the *NN* force. Fourth, both three-body force models appear to provide too much attraction. That is, within the context of a nonrelativistic nucleon assumption, the triton is overbound. This conclusion is, however, strongly dependent upon the value of Λ chosen for the π-nucleon form factor, as one can see from Ref. 7.

Details of the calculation along with results for other properties of ^{3}H and ^{3}He will be reported elsewhere, as will *Nd* scattering-length results. However, it is worth noting that our calculations indicate that these two three-body–force models do not move one off the "Phillips' curves" reported by Friar *et al.*[17]

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