Renormalizing the Nonrenormalizable

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A perturbatively nonrenormalizable variant of the Gross-Neveu model of Euclidean quantum field theory with bare propagator $p/p^{2-\epsilon}$ is considered. We outline a rigorous argument proving that by appropriate choice of the bare coupling constant the model may be renormalized nonperturbatively, which results in a theory governed at short distances by a non-Gaussian fixed point of the renormalization group.

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During the last decade or so the old views on the consistency of local quantum field theories, based on the perturbative classification into superrenormalizable, renormalizable, and nonrenormalizable cases, have undergone a modification. The development of the renormalization-group (RG) ideas relating the field theories to the fixed points of RG transformations,[1] the discovery of the role of asymptotic freedom,[2] and detailed studies of the ϕ_4^4 model[3] raised the point that perturbative renormalizability need not be a sufficient condition for the existence of a field-theory model. On the other hand it was realized that difficulties with some nonrenormalizable theories may reflect the failure of the perturbative approach rather than inherent inconsistencies.[4] For example, it was discovered that some of the nonrenormalizable models may be renormalized in each order of $1/N$ expansion.[5] In the present paper, we provide a rigorous argument for the thesis that perturbative renormalizability need also not be a necessary condition for the existence of a model of quantum fields. More specifically, we outline a construction[6] of the fermionic model in two Euclidean dimensions with the action

$$S=\int \bar{\psi} i \partial\!\!\!/ (-\Delta)^{-\epsilon/2}\psi - g\int (\bar{\psi}\psi)^2, \tag{1}$$

where ψ stands for a multiplet of $N>1$ Dirac fields. For $\epsilon=0$ this is the (renormalizable, asymptotically free) Gross-Neveu model whose massive version has been recently rigorously constructed.[7,8] For $\epsilon>0$ the model becomes perturbatively nonrenormalizable because of the slower decay of the free propagator $p/p^{2-\epsilon}$ at large momenta. Nevertheless, it can still be treated by similar rigorous RG techniques. We introduce an ultraviolet cutoff Λ, replacing the propagator by

$$\Gamma(1-\epsilon/2)^{-1} p\!\!\!/ \int_{\Lambda^{-2}}^{\infty} d\alpha\, \alpha^{-\epsilon/2} e^{-\alpha p^2} \underset{\Lambda\to\infty}{\longrightarrow} p\!\!\!/ /p^{2-\epsilon}. \tag{2}$$

The aim is to choose g_Λ in such a way that a (nontrivial) limit $\Lambda\to\infty$ of the interacting theory exists. The question is studied by consideration of effective lower-energy actions $S_{\tilde\Lambda}^{\rm eff}$ obtained from the bare one S_Λ by lowering the cutoff via integration over the part of the field corresponding to the high-momentum propagator:

$$\hat{\mathscr{T}}_{\Lambda\tilde\Lambda}(p)=\Gamma(1-\epsilon/2)^{-1} p\!\!\!/ \int_{\Lambda^{-2}}^{\tilde\Lambda^{-2}} d\alpha\, \alpha^{-\epsilon/2} e^{-\alpha p^2}. \tag{3}$$

The continuum-limit problem now becomes the question about the existence of the $\Lambda\to\infty$ limits of $S_{\tilde\Lambda}^{\rm eff}$ (or their versions with source terms added to S_Λ) for arbitrary but fixed $\tilde\Lambda$. For easier comparison, one usually rescales the actions to the unit-cutoff regime by introducing the "Hamiltonians"

$$H_\Lambda(\psi)=S_\Lambda(\Lambda^{(1+\epsilon)/2}\psi(\Lambda\,\cdot)), \tag{4}$$

$$H_{\tilde\Lambda}^{\rm eff}(\psi)=S_{\tilde\Lambda}^{\rm eff}(\tilde\Lambda^{(1+\epsilon)/2}\psi(\tilde\Lambda\,\cdot)). \tag{5}$$

H_Λ and $H_{\tilde\Lambda}^{\rm eff}$ are related by the action of the semigroup of renormalization-group transformations: for example for $\Lambda=L^n\tilde\Lambda$,

$$H_{\tilde\Lambda}^{\rm eff}=R^n H. \tag{6}$$

The transformation R is defined by

$$RH(\psi)=H^0(\psi)-\ln\int \exp[-H^I(L^{-(1+\epsilon)/2}\psi(L^{-1}\cdot)+\xi)-\tfrac{1}{2}\langle\bar\xi|\mathscr{T}_{1L^{-1}}^{-1}|\xi\rangle]D\bar\xi D\xi+\text{const}, \tag{7}$$

where $H^0(=H_\Lambda|_{g_\Lambda=0})$ is the free Hamiltonian and $H^I=H-H^0$. The propagator of ξ in Eq. (7) possesses both an ultraviolet and infrared cutoff and if H^I is local (as is H^I_Λ) and has small couplings, the perturbation expansion for RH in powers of H^I converges (a special feature of fermionic cutoff functional integrals[7]). This remains still true for H only approximately local (as is RH produced from a local H) and allows us to introduce a space $\mathscr{M}$ of Hamiltonians such that for H^I in a neighborhood of zero (the Gaussian fixed point of R) in $\mathscr{M}$, $(RH)^I$ is given by the convergent perturbation series in H^I and belongs again to $\mathscr{M}$. More specifically, we consider H^I [with the Euclidean, $U(N)$, and chiral symmetries] given by

$$H^I(\psi)=z\int i\partial\bar{\psi}-g\int(\bar{\psi}\psi)^2+\sum_{m=1}^{\infty}\sum_{(A_1,\ldots,A_{2m})}\int dx_1\cdots dx_{2m}\,\tilde{H}^{2m}(x_1,\ldots,x_{2m},A_1,\ldots,A_{2m})\times:\prod_{i=1}^{2m}\Psi_{A_i}(x_i):_{\leq 6}=(z,g,\tilde{H}), \tag{8}$$

where $\Psi=(\Psi_A)=(\bar{\psi},\psi,\partial\bar{\psi},\partial\psi)$ is the multiplet of fields and their first derivatives and $::_{\leq 6}$ Wick orders with respect to H^0 the terms of order ≤ 6 in Ψ. The $m=1$ term must have both fields differentiated, and $m=2$ at least one, as the leading local terms have been singled out in Eq. (8). Defining

$$\|\tilde{H}^{2m}\|=\sum_{(A_1,\ldots,A_{2m})}\int dx_2\cdots dx_{2m}|\tilde{H}^{2m}(x_1,\ldots,A_{2m})|\exp[\mathscr{L}(x_1,\ldots,x_{2m})], \tag{9}$$

where $\mathscr{L}(x_1,\ldots,x_{2m})$ is the length of the shortest tree on $x_1,\ldots,x_{2m}$, we take $\mathscr{M}$ as the space of H^I of Eq. (8) with

$$\|H^I\|=\sup(|z|,|g|,\|\tilde{H}^{2m}\|g_0^{-\gamma(m)})<\infty, \tag{10}$$

where g_0 is a small constant and $\gamma(1,2)=0$, $\gamma(m)=1+(m-4)/10$ for $m\geq 3$. The choice of $\mathscr{M}$ is obviously motivated by the properties of RH_Λ for small g_Λ.

As R is given by a convergent perturbation expansion in the neighborhood of zero in $\mathscr{M}$, it is easy to establish its leading behavior. For example, in the leading orders we have

$$z\to L^{-\epsilon}z+\gamma_2g^2+\ldots,$$
$$g\to L^{-2\epsilon}g-\beta_2g^2+\ldots, \tag{11}$$

where

$$\gamma_2=\frac{2N-1}{2\pi^2}\ln L+O(\epsilon),$$
$$\beta_2=-\frac{2(N-1)}{\pi}\ln L+O(\epsilon). \tag{12}$$

Neglecting other terms, we obtain a fixed point of Eqs. (11) at

$$z_0^*=\frac{2N-1}{2(N-1)^2}\epsilon+O(\epsilon^2),$$
$$g_0^*=\frac{\pi}{N-1}+O(\epsilon^2). \tag{13}$$

It bifurcates from the Gaussian one at $\epsilon=0$. It is a standard job to establish, using the contractive mapping principle, the existence of a true non-Gaussian fixed point,

$$H^*=H_0^*+O(\epsilon^2), \tag{14}$$

for small ϵ and to study the linearized RG transformation at H^* which appears to have L^{ν_1} and L^{ν_2} with $\nu_1=2\epsilon+O(\epsilon^2)$, $\nu_2=-\epsilon+O(\epsilon^2)$ as the biggest eigenvalues. An easy inductive argument allows us to prove that the line of bare theories $(0,g,0)$ intersects the stable manifold of H^* (of codimension 1) at $g_c=g^*+O(\epsilon^2)$ whereas the unstable one (of dimension 1) connects H^* to the Gaussian fixed point; see Fig. 1.

Now, it is easy to guess the right choice of the bare couplings which leads to a nontrivial continuum limit. Take bare Hamiltonians H_n with

$$g_n=g_c-L^{-n\nu_1}(g^*-g) \tag{15}$$

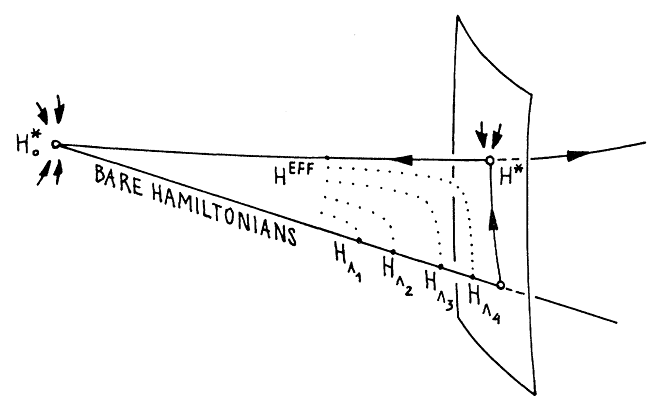

FIG. 1. The renormalization-group flow between the Gaussian (H_0^*) and the non-Gaussian (H^*) fixed points.

and $g < g^*$. Then the flow of H_n under R^n (first along the stable manifold, then along the unstable one) can be easily followed and the desired limit

$$\lim_{n\to\infty} R^{n-k} H_n = H_k^{\mathrm{eff}} \tag{16}$$

lying on the unstable manifold between the fixed points can be easily established by simple geometric analysis in $\mathscr{M}$. Also

$$\lim_{k\to\pm\infty} H_k^{\mathrm{eff}} = \begin{cases} H^* \\ H^0 \end{cases} . \tag{17}$$

g is approximately the quartic coupling of H_0^{eff}. Notice that Eq. (15) corresponds to the choice

$$g_\Lambda = \Lambda^{-2\epsilon}[g_c - (\mu/\Lambda)^{\nu_1}(g^* - g)] \tag{18}$$

of the bare coupling constant.

As Eq. (17) indicates, the continuum-limit Euclidean field theory constructed in this way has short-distance behavior governed by H^* and massless asymptotically free long-distance behavior (the massive theory can be also constructed for $g^* - g$ of both signs). We expect to be able to extend this result to the Gross-Neveu model in three dimensions for big N. Although somewhat academic (the ϵ theory lacks physical positivity because of the nonpolynomial nature of its inverse propagator), the present example should convince us that nonrenormalizable models of field theory governed at short distances by non-Gaussian RG fixed points may be consistent. Whether Nature makes use of such a possibility in the ultrahigh-energy region remains to be seen.

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