Sensitive Dependence on Parameters in Nonlinear Dynamics

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Two qualitatively different types of dynamical behavior can be so tightly interwoven that it becomes impossible to predict when a small change in parameters will cause a change in qualitative properties. For quadratic mappings of the interval, for example, the chaotic parameter values form a Cantor set of positive measure, broken up by periodic intervals. This set can be described by a global scaling law, which makes it possible to form a good estimate of the fraction of chaotic parameter values. Sensitive dependence on parameters occurs when the scaling exponent $\beta < 1$. β is conjectured to display universal behavior.

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There are many respects in which nonlinear dynamical systems can behave in complicated, unpredictable ways. Perhaps the most famous is *chaos,* in which exponential amplification of errors makes it impossible to predict detailed evolution into the far future. When chaos occurs, the only alternative is to resort to statistical prediction. Knowledge of an average quantity that remains constant in time or varies in a known manner may give enough predictive power to be sufficient for many purposes.

As first pointed out by Lorenz, however, statistical averages are not always computable. In his 1964 paper "on determining the climate from the governing equations," he demonstrates that statistical properties can be unstable under variations in parameters.[1] Near a bifurcation, a small change can induce a dramatic change in the qualitative properties of the motion. When the bifurcation set is simple, this is unlikely, but if the bifurcation set is a more complicated object, such as a Cantor set, such behavior may be common.

For example, consider a quadratic mapping of the interval, such as

$$x_{k+1} = f_r(x_k) = r(1-2x_k^2). \tag{1}$$

From a theorem due to Singer,[2] any given value of r generates a unique attractor that determines the asymptotic behavior of almost all initial conditions in $-1 < x < 1$. For convenience, refer to values of r with a chaotic attractor as simply *chaotic parameters,* and call an interval in r whose attractor is a given periodic orbit a *periodic interval.* For maps of this type, stable periodic behavior is widely believed to be dense, i.e., between any two chaotic parameter values there is always a periodic interval. Furthermore, the statistical properties in the periodic intervals are dramatically different from those of nearby chaotic parameters. Thus statistical properties can vary wildly as a function of r, and the presence of even very small amounts of external noise or roundoff can result in dramatic errors in computed values.

This complicated behavior has caused a great deal of confusion. An illustration is provided by the history of a long-standing question in the study of chaos: What fraction of parameter values are chaotic? On the basis of the presumed denseness of the periodic intervals, many early workers conjectured that chaos occurred only on a set of zero measure. Numerical work, however, seemed to indicate the contrary.[1,3,4] Shaw, for example, computed the Lyapunov exponent as a function of r by numerically estimating

$$\lambda = \frac{1}{N}\sum_{k=0}^{N-1} \ln \frac{df}{dx}(x_k) \tag{2}$$

at many values of r.[4] Beyond the first period-doubling accumulation, most of the computed values of λ are positive, indicating chaos, but there are also many sharp dips below zero, corresponding to periodic intervals. As the calculation is done in increasing detail, more and more of these dips are seen as additional periodic intervals are resolved.[5] It remained unclear whether increasing resolution would eventually cause the periodic intervals to consume the majority (or even all) of the chaos. There is now a rigorous proof stating that the chaotic parameter values do in fact have positive measure,[6] but this result gives no information about the actual fraction of parameters leading to chaos.

These questions can be resolved by examining the scaling properties of periodic orbits.[7] The resulting power law makes it possible to estimate the probability that a change of parameters will lead to a change in qualitative behavior. As the size of the perturbation becomes smaller, there is a well-defined sense in which it becomes increasingly less likely to induce bifurcations. This scaling law can also be used to give a precise estimate for the fraction of parameter values that lead to chaos.

Before presenting the numerical results, it is necessary to introduce some definitions, assumptions, and terminology. First, for maps such as (1), assume that chaotic and periodic motion are the only classes of dynamical behavior with positive Lebesgue measure in r. (Other possibilities are, for example, period-doubling accumulation points, but these seem unlikely to consume a finite fraction of the parameter interval.)

Assuming that the periodic parameters are indeed dense implies that the chaotic parameters form a Cantor set. Unlike the Cantor sets that are most popular these days, these have positive Lebesgue measure, and are thus "fat." (Mandelbrot[8] discusses some examples of fat Cantor sets, and some of their more general counterparts, e.g., what he calls "dusts with positive volume.") To see the distinction, consider the classic Cantor set, constructed by deleting the central third of an interval and then deleting the central third of each remaining subinterval *ad infinitum.* It has zero Lebesgue measure and fractal dimension log2/log3. Compare this to a fattened version, obtained by deleting instead the central $\frac{1}{3}$, then $\frac{1}{9}$, then $\frac{1}{27}$, etc., of each piece. The resulting set is topologically equivalent, but the holes decrease in size sufficiently fast so that the resulting limit set has positive Lebesgue measure and fractal dimension 1.

For fat Cantor sets, dimension does not properly characterize fractal properties, since the dimension is an integer. Nonetheless, like all fractals, apparent size depends on the scale of resolution. To make this more explicit, for a Cantor set on [0,1], let $h(\epsilon)$ be the total size of all holes whose width is greater than or equal to ϵ. Define the ϵ *coarse-grained measure* as $\mu(\epsilon) = 1-h(\epsilon)$. By definition $\mu(\epsilon)$ is a nondecreasing function of ϵ. $d\mu(\epsilon)/d\epsilon=0$ for all ϵ except those values of ϵ corresponding to hole sizes, at which $\mu(\epsilon)$ is discontinuous. $\mu(\epsilon)$ is thus a staircase function with steps at each hole size. For nonfractal sets, $\mu(\epsilon)$ reaches its limiting value $\mu(0)$ for $\epsilon > 0$. For fractals, however, this limiting value is reached only when $\epsilon = 0$. For thin Cantor sets, $\mu(0)=0$, whereas for fat ones, $\mu(0)>0$. These ideas can be extended to sets of any dimension, by modifying the definition of $\mu(\epsilon)$ to deal with the generalization of fat Cantor sets, dubbed "fat fractals."[9]

Although $\mu(\epsilon)$ is a staircase function, near $\epsilon = 0$ the steps become small, so that $\mu(0)$ can be approximated by a smooth function. For quadratic maps of the interval, I conjecture that $\mu(\epsilon)$ asymptotically scales as a power law in the limit as $\epsilon \to 0$,

$$\mu(\epsilon) \approx \mu(0) + A\epsilon^{\beta}. \tag{3}$$

$\mu(0)$ is the true measure of chaotic parameters, and the value of A depends on the units used. As discussed later, similar scaling behavior also occurs for chaotic orbits in twist maps, and for the ergodic parameter values of subcritical circle maps.[9,10]

The most essential number in Eq. (3) is the exponent β of Eq. (2), dubbed the *fatness exponent.* It can be defined more precisely as

$$\beta = \lim_{\epsilon \to 0} \frac{\log[\mu(\epsilon)-\mu(0)]}{\log\epsilon}. \tag{4}$$

By definition $\beta \geq 0$. For nonfractal sets, $\beta = \infty$, since $\mu(\epsilon) = \mu(0)$ for all ϵ less than some fixed value. For fractals, β is finite and provides a means for quantifying fractal properties. For thin fractals, β is equal to the codimension $D-d$, where d is the fractal dimension and D is the dimension of the embedding space.[11] For fat fractals, however, β is independent of d. In a sense, β is more generally useful for discussing fractals than the dimension, since it describes the manner in which the apparent size of visible holes scales with resolution for both thin and fat fractals.

To understand the relevance of this to dynamical systems theory, first consider the more familiar case in which the bifurcation sets are simple surfaces of codimension 1. For a one-parameter family, for example, suppose that bifurcations take place only at a finite number of isolated parameter values. Select a parameter value at random, and make a random variation about it of size ϵ. The fraction of these values with qualitatively different behavior is proportional to ϵ. This remains true in a parameter space of any dimension, provided that the bifurcation surfaces are of codimension 1. In contrast, if the bifurcation surfaces are more complicated and form a fat fractal, the probability of a bifurcation scales as $\epsilon + \epsilon^{\beta}$, and the second term dominates in the limit as $\epsilon \to 0$ unless $\beta > 1$. This prompts the following definition:

Definition.—A dynamical system displays *sensitive dependence on parameters* when its bifurcation set has $\beta < 1$.

Alternatively, for a system driven by external noise, ϵ can be replaced by the amplitude of the noise. The measure of chaos as determined by computing the Lyapunov exponent,[5] for example, will scale as $\epsilon + \epsilon^{\beta}$. Again, the second term dominates unless $\beta > 1$.

To apply this to maps such as Eq. (1), I explicitly computed the boundaries of almost all the periodic intervals larger than a given threshold. This calculation was complicated by the fact that there can be an enormous number of periodic orbits with any given period p, making it difficult to locate each interval by use of straightforward methods. This problem can be avoided by making use of the results of Metropolis, Stein, and Stein[12] (MSS), which enumerate and order the periodic intervals. They begin by assigning every orbit a sequence of R and L values that uniquely characterize it. To do this, start at the critical point $x_c = x$, where $(df/dx)(x) = 0$, and iterate the map, assigning the symbol R for every value to the right of the critical point and L for every value to the left. The resulting sequences are in a one-to-one correspondence with the periodic intervals. Starting with the first sequence, R, corresponding to the smallest value r, and the last sequence *RLLL. . .* , the operation defined by MSS can be recursively applied to generate all sequences corresponding to periodic intervals, in their proper order on the parameter interval r. The numerical experiments of MSS and others indicate that this ordering is universal, i.e., that it applies to all maps in the same universality class.

The stability interval for a given orbit can be found by taking advantage of knowledge of the symbol sequence. The superstable parameter value (at which the derivative of the orbit is zero) can be computed by finding the zero crossing in r of the function

$$g_r(x) = f_r^{-(p-1)}(x) - f_r(x), \tag{5}$$

with $x = 0$, where $f^{-(p-1)}$ is the inverse iterate of f applied $p-1$ times, and depends on the symbol sequence.[13] This method has two significant advantages over the straightforward approach using forward iterates. First, as far as I can determine from numerical studies, every symbol sequence generates a unique zero crossing, eliminating the problem of degeneracy.[14] The second advantage is that iteration of this function is much more stable than iteration of the forward iterate. This can be seen from the following argument: The derivative of the pth iterate of f is of the form $df^p/dx = D_0 D_1 \cdots D_{p-1}$, where D_i is the derivative of the ith iterate, starting at the point closest to the critical point x_c. Typically, D_0 is small, while $D_1, \ldots, D_{p-1}$ are mainly greater than 1, a fact that causes numerical instabilities in the iteration of long orbits. When the iteration is done in the order given by Eq. (5), however, the offending derivatives are inverted, which makes the numerical properties extremely stable.

Once the superstable parameter value is obtained, the stability interval can be found by computing the period-doubling accumulation on one side, and the tangent bifurcation on the other. The period-doubling accumulations are found by generating the appropriate symbol sequence and superstable parameter value for each period doubling and extrapolating to find the accumulation at each step until the extrapolated value converges to an acceptable tolerance. The tangent bifurcations are found by use of a two-dimensional modified Newton method in r and x to find the point where $g_r(x) = 0$ and $(df^p/dx)(x) = 1$. By applying these techniques, it is possible to find the boundaries of any given periodic interval with an accuracy approaching machine precision.

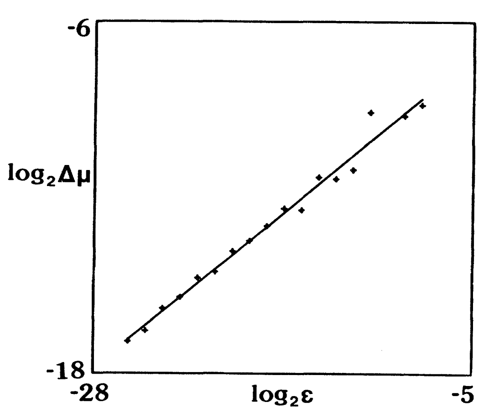

FIG. 1. The logarithm of change in the coarse-grained measure $\log_2[\Delta\mu(\epsilon)]$ plotted against the resolution $\log_2\epsilon$.

With use of this procedure, $\mu(\epsilon)$ was obtained by computing almost all[15] of the roughly 4000 periodic intervals of size greater than 10^{-8}. To check the scaling law postulated in Eq. (4), the logarithm of the change in $\mu(\epsilon)$ was plotted against the logarithm of ϵ, as shown in Fig. 1. This was done for both Eq. (1) and for

$$x_{k+1} = f_r(x_k) = r \sin(\pi x_k). \tag{6}$$

In both cases a good fit was obtained at small values of ϵ. The resulting estimates for $\mu(0)$ and β are shown in Table I. Note that the error bars on $\mu(0)$ are quite small, which gives an accurate estimate for the measure of chaos. The computed values of β are less accurate, but the results clearly indicate that both maps display sensitive dependence on parameters. There is no reason to believe that $\mu(0)$ should be the same for the two maps, and, indeed, the computed difference is well above the experimental error. However, I conjecture that the value of β is universal for a given order of the maximum. The numerical correspondence here is not good enough to be convincing, but I hope to present more detailed numerical computations in a future paper. There is also a well-defined exponential scaling law for μ as a function of the period of the orbits, which will also be presented later.

These results imply that the probability of a qualitative change in behavior due to a change in parameters behaves in a well-defined manner. For example, one might ask the following: In a numerical experiment, what is the likelihood that a parameter setting believed to be chaotic is actually periodic? This can be estimated as $\mu(\epsilon) - \mu(0) = A\epsilon^\beta$. Assuming that A is the order of 1, and taking $\beta \approx 0.45$ and $\epsilon = 10^{-14}$, implies that the odds of a mistake are roughly one in a million. This demonstrates that numerical computations done on quadratic mappings with typical computer precision are generally reliable. Even though there is always a chance that any given result is wrong, the probability

TABLE I. Computed values of $\mu(0)$ and β for Eqs. (1) and (6). $\mu(0)$ is the fraction of chaotic parameter values between the first period-doubling accumulation and $r = 1$, and β is the scaling exponent defined in Eq. (4).

Equation	$\mu(0)$	β
(1)	0.89795 ± 0.00005	0.45 ± 0.04
(6)	0.8929 ± 0.0001	0.45 ± 0.04

of error goes to zero systematically as the precision is improved. If the conjecture of universality is indeed correct, then the scaling exponent is the same for an entire class of systems.

Note that the concept of universality indicated here is substantially different from that originally found by Feigenbaum,[16] since it applies to a set of positive measure rather than just special points such as period-doubling transitions. This suggests that there is a global renormalization theory. Such a theory would extend the notion of metric universality to the entire MSS sequence, synthesizing the work of MSS with that of Feigenbaum.

As already mentioned, these basic ideas can also be applied to other examples. As discussed in Ref. 10, the set of parameter values where quasiperiodic behavior occurs in subcritical circle mappings forms a fat Cantor set, with the power-law scaling of Eq. (3). β varies smoothly as a function of the nonlinearity parameter through the transition to critical behavior, indicating a second-order phase transition from a fat to a thin Cantor set. As discussed in Ref. 9, another application occurs in two-dimensional Hamiltonian mappings. For chaotic orbits, a single initial condition fills out a region with a complicated structure, containing holes associated with the islands of stability surrounding elliptic fixed points. The coarse-grained measure $\mu(\epsilon)$ of these chaotic orbits shows power-law scaling, associated with the global scaling properties of the island chains.

A question that remains open at this point is whether this scaling applies to higher-dimensional systems, such as the Rossler equations or the Henon map. I strongly feel that they do, at least in some approximate sense, and hope that this paper will stimulate work on these systems.

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[14]There is a complication: For prime bands (see Ref. 3) there is a unique zero crossing. For composite bands this function touches zero for each higher-order band, but only actually *crosses* zero at the correct parameter value.

[15]It is possible to arrange the MSS sequences in a binary tree, such that sequences lower down in the tree generally have smaller stable intervals. There are exceptions, but they are rare, and numerical evidence indicates that the fraction of exceptional cases goes to zero as the period increases. This will be discussed in more detail in a future paper.

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