Tricritical Point in Random-Field Ising Model

The numerical investigation of the random-field Is-
ing model by Houghton, Khurana, and Seco[1] has re-
vealed a number of interesting features concerning the
three-dimensional Ising model in a random external field
B_i. Earlier, Aharony[2] had already found that even in a
mean-field approximation the bimodal $[B_i = \pm B]$ and
Gaussian $[P(B_i) \approx \exp(-B_i^2/2B^2)]$ distributions led
to different phase diagrams. (As one increases T at
fixed B, the bimodal distribution has a second-order
phase transition for B in the range $0 < B < 0.44$ which
becomes first order for B in the range $0.44 \leq B < 0.5$.
By way of contrast, in the Gaussian case, in the range
of fields where the latter has a phase transition this
transition is always second order.) In short, the bimo-
dal distribution has a tricritical point at $T = 0.67$,
$B = 0.44$, whereas the Gaussian does not. This is
surprising, in that it is generally believed that only the
average and the variance of random distributions are
significant. Additionally, the detailed studies of
Houghton, Khurana, and Seco[1] show the importance
of dimensionality as well as of the shape of the random
distribution on the thermodynamic properties of these
special random systems.

In this note, I consider the *tri*modal distribution:
Each spin is in a field $B_i = 0, \pm B$. The probability of
$B_i = 0$ is P. If we assume that $\pm B$ are equally likely,
the probability of each is $(1 - P)/2$. (This distribution
allows a physical interpretation as a diluted bimodal
distribution in which a fraction P of the spins are not

exposed to the external field.) Our model mimics the
salient feature of the Gaussian distribution, for which
a significant fraction of the spins are in weak external
fields. Assuming that it is this feature of the Gaussian
which leads to a behavior different from the bimodal
distribution, we may expect the trimodal distribution
also to lose the tricritical point for P greater than some
P_c to be determined.

The principal advantage of the discrete trimodal
over the continuous Gaussian is simplicity. Here, at
least, the mean-field equation of state for the order
parameter σ can be obtained conveniently in closed
form. It is

$$\sigma = [\tanh\beta\sigma][P + (1-P)/(1+R^2)], \tag{1}$$

where

$$R \equiv \sinh^2\beta B/\cosh^2\beta\sigma. \tag{2}$$

When $P = 0$, one recovers the bimodal model original-
ly solved by Aharony.[2] (It has the above-mentioned
tricritical point.) At $P = 1$, arbitrary B, we recover the
usual (zero-field !) mean-field equation, the prototype
of a second-order phase transition.[3] It remains to
determine P_c separating the two behaviors.

I have approached the problem from the following
point of view[4]: Letting $\sigma \to 0$ in (1) and (2) yields a
generally two-valued, continuous function $T_c(B)$,
given implicitly by the equation

$$T_c = P + (1-P)/\cosh^2(B/T_c) \tag{3}$$

(with $k_B = 1$). Such a function must have at least one
point at which $dT_c/dB = \infty$. Denote it by (T_0, B). It
is a matter of simple algebra to obtain

$$(T_0/2)\{1 + (T_0 - P)/[1 - T_0 + (1-P)^{1/2}(1-T_0)^{1/2}]\}$$
$$= (T_0 - P)\ln\{[(1-P)^{1/2} + (1-T_0)^{1/2}]/(T_0 - P)^{1/2}\}, \tag{4}$$

and a matter of simple numerics to analyze it. The
result: Solutions exist only for P within the range
$0 \leq P < 0.25$. At $P_c = 0.25$ the vestigial tricritical
point hangs up at $T_c = 0.44$ and $B_c = 0.57$. [For P
≥ 0.25 the phase transition becomes second order at
temperature $T_c(B)$, a rapidly decreasing function of B
which vanishes once B exceeds a critical value depend-
ing on P.] Because a good approximation to the
Gaussian has $P \approx \frac{1}{3}$ we believe that our analysis pro-
vides independent confirmation of the lack of tricriti-
cal points for nearest-neighbor Gaussian models in
$d \geq 6$ and in mean-field theory.

Finally, in light of Ref. 1, it would be interesting to
see whether for a nearest-neighbor model in 3D, the
relevant P_c has been boosted to a value $\gtrsim \frac{1}{3}$. This
suggests a need for numerical studies of the T, P, B
phase diagram of the dilute bimodal, which we have

called the *trimodal*, distribution in 3D.

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[2]A. Aharony, Phys. Rev. B **18**, 3318 (1978).
[3]D. Mattis, *The Theory of Magnetism* (Springer-Verlag,
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[4]See also S. Galam and J. L. Birman, Phys. Rev. B **28**,
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