Quantum Dynamics and Statistics of Vortices in Two-Dimensional Superfluids

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(Received 11 October 1985)

The phase change of the wave function of a superfluid film as a vortex moves around a closed path counts the number of superfluid particles enclosed by that path. This result is used to investigate whether such vortices obey "fractional statistics." We conclude that this is not the case in compressible superfluids such as ^{4}He films.

PACS numbers: 03.40.Gc, 05.30.–d, 67.40.Vs

The possibility that elementary excitations in two-dimensional (2D) systems may obey nonstandard quantum statistics has aroused recent interest.[1,2] The intriguing suggestion that vortex topological excitations in superfluid films might exhibit such effects has been made recently by Chiao and co-workers.[3] However, these authors' conclusions are based on nonmicroscopic arguments; here we present a microscopic analysis of this question which suggests a quite different result.

A superfluid film may be considered as quantum mechanically two-dimensional if the fluid is bound in a quantum well at a substrate surface, and the temperature is sufficiently low that there is only zero-point motion normal to the surface. This situation may be more or less realized in thin ^{4}He films with surface coverage of order of a monolayer; however, thin superconducting films do not appear able to meet this criterion.

Fetter[4] has pointed out that since the phenomenological hydrodynamic equations of motion of superfluid vortices in the absence of dissipation can be described in a Hamiltonian formulation, they can be quantized. Elementary vortex excitations in a mass-m Bose-particle superfluid carry quantized circulation $\pm h/m$.[5] The superfluid velocity field of a system of such vortices with circulation γ_ν and centers $\mathbf{R}_\nu$ is

$$\mathbf{v}(\mathbf{r}) = (1/2\pi)\hat{\mathbf{z}}\times\nabla\left(\textstyle\sum_\nu \gamma_\nu \ln|\mathbf{r}-\mathbf{R}_\nu|\right), \tag{1}$$

where $\hat{\mathbf{z}}$ is the direction normal to the surface. A vortex moves at the local superfluid velocity at its center:

$$\dot{\mathbf{R}}_\nu = \frac{1}{2\pi}\sum_{\nu'\neq\nu}\gamma_{\nu'}\frac{\hat{\mathbf{z}}\times(\mathbf{R}_\nu-\mathbf{R}_{\nu'})}{|\mathbf{R}_\nu-\mathbf{R}_{\nu'}|^2}. \tag{2}$$

If the superfluid is modeled as an incompressible fluid, the energy of the vortex configuration is purely kinetic, and up to a constant term is given by

$$H = -\frac{\rho m}{2\pi}\sum_{\nu<\nu'}\gamma_\nu\gamma_{\nu'}\ln(|\mathbf{R}_\nu-\mathbf{R}_{\nu'}|/\xi), \tag{3}$$

where ρ is the superfluid surface particle density, and ξ is a length scale characterizing the vortex core; (3) is valid for vortex separations large compared to ξ.

The equation of motion (2) may be obtained[4] by use of (3) as the Hamiltonian function by introduction of commutation relations

$$[R^i_\nu, R^j_{\nu'}] = iq_\nu\delta_{\nu\nu'}l^2\epsilon^{ijk}\hat{z}^k, \tag{4}$$

where $q_\nu = \gamma_\nu m/h = \pm 1$ is the topological charge of the vortex, and we have defined the characteristic length $l = (2\pi\rho)^{-1/2}$; $2\pi l^2$ is the *surface area per particle of the superfluid film.* [If (4) is derived from the London equations for a superconducting film,[4] $2\pi l^2$ is the surface area per Cooper pair.] The commutation relation (4) is formally identical to that to the "guiding-center" coordinates of a 2D electron moving in a cyclotron orbit in a magnetic field, where l is the "magnetic length," i.e., $2\pi l^2$ is the surface area per magnetic flux quantum h/e.

The phenomenological commutation relation (4) strongly suggests the following conjecture: *The phase change of the superfluid wave function as a vortex is moved adiabatically around a closed path includes a term $2\pi q$ times the mean number of superfluid particles enclosed by the path.*[6] This is the direct analog of the corresponding result for a charged particle moving in a magnetic field. The relation (4) also implies the uncertainty principle that a vortex center cannot be localized within an area smaller than the mean area per particle, thus reflecting the underlying particulate nature of the superfluid.

The above phenomenological arguments can be supported by a microscopic argument. In a noninteracting Bose condensate, the ground-state wave function Ψ^0

and the one-vortex–state wave functions $\Psi^{\pm 1}$ (centered at the origin) are given by product wave functions $\Psi^m = \prod_i \psi^m(\mathbf{r}_i)$, where $\psi^m(\mathbf{r})$ is the lowest-energy single-particle wave function with surface angular momentum $L^z = m/\hbar$. Since (away from the boundaries of the system) $\psi^{\pm 1}(\mathbf{r})/\psi^0(\mathbf{r}) \propto x \pm iy$, the one-vortex state of a large system can be written as $\Psi^{\pm 1} \propto A^{(\pm)}(\mathbf{0})\Psi^0$, where

$$A^{(\pm)}(\mathbf{R}) = \prod_i [(x_i - X) \pm i(y_i - Y)], \tag{5}$$

and $(x_i, y_i) = \mathbf{r}_i$ and $(X, Y) = \mathbf{R}$ are the surface coordinates of the superfluid particles and the vortex center. This form suggests the following *Ansatz* for the (dilute) multivortex state of the *interacting* system:

$$\Psi \propto \prod_\nu [A_\nu^{(q)}(\mathbf{R}_\nu) f(\mathbf{R}_\nu)] \Psi^0, \tag{6}$$

where Ψ^0 is the true ground state, and $f(\mathbf{R}) = \prod_i g(|\mathbf{r}_i - \mathbf{R}|/\xi)$, with $g(r)$ a real function, is a factor allowing for relaxation of the particle density profile of the vortex in the interacting superfluid: $f(r) \sim \text{const}/r$ for large r as the density becomes uniform far from the vortex. This *Ansatz* was proposed for the single vortex by Feynman[5]; Fetter[7] has studied this and improved forms, and also considered product wave functions for multivortex states in the Gross-Pitaevski formulation.[8]

The superfluid velocity field is given by

$$\mathbf{v}(\mathbf{r}) = \langle \Psi | \mathbf{j}(\mathbf{r}) \Psi \rangle / \langle \Psi | \rho(\mathbf{r}) | \Psi \rangle, \tag{7}$$

where $\rho(\mathbf{r}) = \sum_i \delta^2(\mathbf{r} - \mathbf{r}_i)$ and $\mathbf{j}(r) = \sum_i \mathbf{p}_i \delta^2(\mathbf{r} - \mathbf{r}_i)/2m + \text{H.c.}$ are the superfluid density and current operators. For the wave function (6), $\mathbf{v}(\mathbf{r})$ is easily evaluated exactly by use of the commutation relations of $\mathbf{j}(\mathbf{r})$ and $A^{(\pm)}(\mathbf{R})$, plus the time-reversal invariance property $\Psi^{0*} \propto \Psi^0$ of the ground state: $\mathbf{v}(\mathbf{r})$ for (6) is identical to the hydrodynamical expression (1) with $\gamma_\nu = q_\nu h/m$. Similarly, the energy of Ψ relative to Ψ^0 can be obtained as (3) plus core-energy corrections.

The angular momentum operator L^z acting on (6) has the result

$$L^z \Psi = \hbar \sum_\nu (N q_\nu + i\hat{\mathbf{z}} \cdot \mathbf{R}_\nu \times \partial/\partial \mathbf{R}_\nu) \Psi, \tag{8}$$

where N is the number of superfluid particles. Note that if a net vorticity is present (i.e., $\sum q_\nu \neq 0$), L^z is divergent in the limit of an infinite fluid ($N \to \infty$).

The change of phase of Ψ as a vortex coordinate $\mathbf{R}_\nu$ is moved around a closed path Γ is given by Berry's phase[9]:

$$\Delta\Phi_\nu(\Gamma) = i \oint_\Gamma d\mathbf{R}_\nu \cdot \langle \Psi (\partial/\partial \mathbf{R}_\nu) \Psi \rangle, \tag{9}$$

where Ψ is assumed normalized. For the wave function (6), the form of this integral is essentially the same as that considered by Arovas, Schrieffer, and Wilczek[10] in connection with the fractional quantized Hall effect; repeating their manipulations, we have

$$\Delta\phi_\nu(\Gamma) = q_\nu \int d^2r \oint_\Gamma \hat{\mathbf{z}} \cdot d\mathbf{R}_\nu \times \nabla(\ln|\mathbf{r} - \mathbf{R}_\nu|) \rho(\mathbf{r}; \{\mathbf{R}_{\nu'}\}), \tag{10}$$

where $\rho(\mathbf{r}; \{\mathbf{R}_{\nu'}\}) = \langle \Psi | \rho(\mathbf{r}) | \Psi \rangle$. To proceed, we write $\rho(\mathbf{r}; \{\mathbf{R}_{\nu'}\}) = \rho^0(\mathbf{r}) + \delta\rho(\mathbf{r} - \mathbf{R}_\nu)$, where $\rho^0(\mathbf{r}) = \rho(\mathbf{r}; \{\mathbf{R}_{\nu'}, \nu' \neq \nu\})$ is the expectation value of the density operator in the *absence* of the vortex being moved, and $\delta\rho(\mathbf{r} - \mathbf{R}_\nu)$ is the change due to its presence. In the limit $|\mathbf{R}_\nu - \mathbf{R}_{\nu'}| \to \infty$, $\nu \neq \nu'$, $\delta\rho(\mathbf{r})$ becomes a rotationally invariant function $\delta\rho_0(r)$, the density depletion profile of an isolated vortex. Since $\rho^0(\mathbf{r})$ is independent of $\mathbf{R}_\nu$, Stokes's theorem can be applied to part of (10) to give

$$\Delta\phi_\nu(\Gamma) = 2\pi q_\nu \int_\Gamma d^2r\, \rho^0(\mathbf{r}) + q_\nu \oint_\Gamma \hat{\mathbf{z}} \times d\mathbf{R}_\nu \cdot \int d^2r\, (\mathbf{r}/r^2)[\delta\rho(\mathbf{r}) - \delta\rho_0(r)]. \tag{11}$$

The area integral of the first term is over the area enclosed by Γ; in the second term, the rotationally invariant part of $\delta\rho(\mathbf{r})$ does not contribute, and $\delta\rho_0(r)$ for the isolated vortex can be subtracted.

The expression (11) for $\Delta\phi_\nu(\Gamma)$ consists of two parts: a part $2\pi q_\nu$ times the mean number of superfluid particles (in the *absence* of the vortex at $\mathbf{R}_\nu$) enclosed by Γ, plus residual vortex-interaction terms that become small in a dilute-vortex limit, provided no point on the path Γ comes to a vortex core. This analysis of the Berry phase of a microscopic *Ansatz* wave function is in agreement with the result conjectured from the phenomenological commutation relation (4).

The expression (10) indicates that the effective Lagrangean describing vortex dynamics will depend on vortex velocities $\dot{\mathbf{R}}_\nu$ through a linear term

$$L_1 = \hbar \sum_\nu q_\nu \hat{\mathbf{z}} \times \dot{\mathbf{R}}_\nu \cdot \int d^2r\, \rho(r) \nabla \ln|\mathbf{r} - \mathbf{R}_\nu|. \tag{12}$$

The integral $\int L_1 dt$ for a closed vortex path $\mathbf{R}_\nu(t)$ is equal to $\hbar q_\nu$ times the integral of $\rho(\mathbf{r})$ over the area enclosed by the vortex path, and thus L_1 is precisely the term that will contribute the Berry phase obtained here to the action integral for such a process.

An answer to the question of vortex statistics can now be given. If, like Chiao and co-workers,[3] we consider the phenomenological model of an incompressible fluid where $\rho(\mathbf{r}) = \rho_0$ for $|\mathbf{r} - \mathbf{R}_\nu| > \xi$, then $\Delta\phi_\nu = 2\pi q_\nu(\rho_0 A - N_\nu \pi \rho_0 \xi^2)$, where A is the area enclosed by Γ and N_ν is the number of enclosed vortices (of either sign). This implies that the vortices behave like "fractional statistics" objects[1] with topological parameter $\theta=\pi\delta n$, where $\delta n=\pi\rho_0\xi^2$ is the number of superfluid particles excluded from the vortex core. Note that in the phenomenological model, this is a continuously variable parameter.

In their recent Letters,[3] Chiao and co-workers attempted to derive a θ parameter from consideration of the spectrum of eigenvalues of an operator generating rotations of the vortex coordinates. In fact, the generator of rotations is undetermined up to a constant, and an absolute expression for L^z can only be obtained from a microscopic calculation in terms of the background superfluid; in the case of the superfluid, there is a term $N\hbar\Sigma q_\nu$ that diverges in the limit of the infinite fluid. The argument of Ref. 3 is based upon a separation of the total angular momentum into a term generating rotations of the center of vorticity, and a residual term depending only on the relative coordinates of vortices. Despite the assertions of Ref. 3, it does not appear to us that there is any unambiguous way to partition the *constant* term between these two parts of the total angular momentum. With the choice made in Ref. 3, the relative-motion angular momentum of a two-vortex system has eigenvalues $\hbar(n+\frac{1}{2})$, which Chiao and co-workers suggest implies nonstandard statistics. However, because of the inherent ambiguity involving the constant term in a separation into "center of vorticity" and "relative" angular momenta, we do not believe the nonmicroscopic arguments of Ref. 3 for a "universal vortex-number dependent statistics" to be valid, and suggest that such questions can only be decided by microscopic calculation of the Berry phase.

The phenomenological assumption of incompressibility has serious shortcomings when applied to microscopic models of superfluids, and the above analysis of statistics fails in the case of ^{4}He films because they are *compressible:* If P is the (surface) pressure $-[\partial E/\partial A]_N$ and $\kappa=\rho\, dP/d\rho$ is the (surface) compressibility, the pressure gradient $dP/dr=-\hbar^2\rho/mr^3$ due to the circulation around the vortex leads to an asymptotic superfluid density profile $\rho(r)=\rho(\infty)[1-(\xi/r)^2]$, where $\xi=(\hbar^2\rho/2m\kappa)^{1/2}$ is the "healing length" of the superfluid. The net deficit of superfluid particles within a radius r of the vortex center does *not* converge to a finite value θ/π as $r\rightarrow\infty$. There is thus *no* "dilute limit" in which the quantum dynamics of vortices in a compressible superfluid can be replaced by that of point objects obeying "θ statistics."[1]

If the interaction potential between superfluid particles falls off as $1/r^2$ or slower, the fluid is incompressible at long wavelengths, and a "neutralizing background" potential is required for the existence of the thermodynamic limit. For a potential falling off more slowly than $1/r^2$, screening prevents the accumulation of any net superfluid deficit on the vortex core, and so $\theta=0$. However, in the special case of repulsive interactions falling off as g/r^2 at large distances, the potential of a finite superfluid deficit $\theta/\pi=\hbar^2/2mg$ asymptotically balances the pressure gradient of the circulation, and a description of vortices in such a system as θ-statistics objects should be possible. However, we emphasize that this special type of long-range force is *not* present in physically realized superfluid films.

In summary, we have obtained the phase change of the 2D superfluid wave function that accompanies motion of a vortex around a closed path: It is 2π times the mean number of superfluid particles enclosed by that path. In the case of an *incompressible* 2D superfluid, this implies that the topological parameter θ would be π times the mean depletion of superfluid particles from the vortex core. However, we must stress that physical ^{4}He films constitute a *compressible* superfluid, and there is *no* limit in which their vortices can be described as obeying fractional statistics. Our results thus contradict recent claims of Chiao and co-workers.[3] They may also provide a basis for a semiclassical quantization of vortex dynamics.[11]

This work was carried out at the Aspen Center for Physics, whose hospitality is gratefully acknowledged, and was supported in part by National Science Foundation Grants No. DMR-8405347 and No. PHY-8405648. One of us (F.D.M.H.) is also the grateful re- an Alfred P. Sloan Foundation Fellowship. The other of us (Y.S.W.) has greatly benefitted from numerous discussions with Professor R. Y. Chiao, who drew his attention to this interesting problem.

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[11]The coordinates $\mathbf{R}_\nu$ in (6) are classical c-number variables. In the fully quantum description implied by the *operator* relation (4), (6) would be multiplied by a vortex-coordinate wave function $\overline{\Psi}(\{\mathbf{R}_\nu\})^*$, and the vortex coordinates integrated out. Alternatively, quantization via the path-integral approach could be carried out, with use of (12) for the term in the action dependent on the vortex velocities.