Propagative Phase Dynamics for Systems with Galilean Invariance

P. Coullet[(a)]
Laboratoire de Physique de la Matière Condensée, 06034 Nice, France

and

S. Fauve
Groupe de Physique des Solides de l'Ecole Normale Supérieure, 75005 Paris, France
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We present the nonlinear phase equations describing the stability of a one-dimensional periodic pattern, with mean flow effect due to *Galilean invariance.* We show that the phase dynamics is second order in time, and could lead to an oscillatory instability of the pattern.

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In many nonequilibrium systems driven by spatially homogeneous forcing there is a transition from a uniform state to one varying periodically in space. Long-wavelength modes are very general features of such dissipative structures, and can be traced to continuous broken symmetries. The case of translational invariance has been investigated by several authors who have studied pattern dynamics in reaction-diffusion equations,[1,2] in the Rayleigh-Bénard instability,[3] and in the Taylor-Couette instability.[4] We have pointed out that another phase variable exists as a result of the broken Galilean invariance.[5] We show with the help of a simple model that the phase dynamics is no longer diffusive when the *Galilean invariance* is taken into account, and is associated with a *codimension-two singularity* that leads to *propagative modes.*

We consider a supercritical bifurcation that leads to a one-dimensional pattern described by a Landau-Ginzburg–type equation[6]

$$A_t = A + A_{xx} - |A|^2 A, \tag{1}$$

where $A(x,t)$ is a complex amplitude that slowly modulates a pattern of wave number q_0. Dimensionless units are used in Eq. (1) and the subscripts denote partial derivatives.

Let us suppose that the full set of equations describing our problem is Galilean invariant, and involves a velocity field $B(x,t)$.[7] It follows that a constant velocity field B along the x axis is undamped, and correspondingly weakly damped for slow inhomogeneities. This mean drift B is thus another dangerous mode, i.e., a mode with nearly zero growth rate, that one expects to be coupled with A. This occurs in Rayleigh-Bénard convection, where such a large-scale velocity field is driven by inhomogeneities of the convective pattern, and in turn tends to convect the roll pattern.[8] The correct description of the instability onset therefore requires two coupled order parameters, A and B.

In our one-dimensional model Eq. (1) is modified and becomes

$$A_t = A + A_{xx} - |A|^2 A - iq_0 AB - BA_x. \tag{2}$$

The meaning of the new terms, as noted in Ref. 8, is the advection of the pattern by the velocity field B. Indeed, if the pattern of wave number q_0 is advected at a constant speed B_0, one observes a temporal oscillation at frequency $q_0 B_0$ in the laboratory frame.

With the use of symmetry arguments (e.g., translational and Galilean invariances, and space-reflection symmetry) the equation for B reads

$$B_t = \lambda B_{xx} + \sigma |A|^2_x - BB_x, \tag{3}$$

where λ measures viscous diffusion and σ is the strength of the coupling; we note that B is only generated by pattern inhomogeneities. We first study the one-dimensional pattern dynamics described by Eqs. (2) and (3), and show the basic mechanism of interaction between the neutral modes associated with translational and Galilean invariances. We then apply the same technique for the specific example of thermal convection, and derive a nonlinear evolution equation for the oscillatory instability.[9]

The family of stationary solutions of Eqs. (2) and (3),

$$A_0(x) = Qe^{iqx}, \quad B_0 = 0, \tag{4}$$

with $Q^2 = 1 - q^2$ and $|q| < 1$, represents perfectly periodic patterns. To investigate their stability we define $a(x,t)$ and $b(x,t)$ by

$$\begin{aligned} A(x,t) &= A_0(x) + a(x,t)e^{iqx}, \\ B(x,t) &= -b(x,t)/(q_0+q)Q, \end{aligned} \tag{5}$$

and linearize Eqs. (2) and (3) in a and b. For spatially constant a and b, we find

$$a_t = -2Q^2 \operatorname{Re} a + ib, \quad b_t = 0, \tag{6}$$

where Rea denotes the real part of a. Equation (6) can be rewritten as

$$\rho_t = -2Q^2\rho, \quad \begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{7}$$

where $a = \rho + i\phi$ and $b = \psi$; thus, in the long-wavelength limit, we have two neutral modes, ϕ and ψ, respectively related to *translational and Galilean invariances.* Their linear coupling can be understood as follows: The advection of the pattern at a constant speed ψ leads to a spatial phase ϕ that increases linearly in time. Equation (7) represents a codimension-two bifurcation.[10] Two control parameters are usually necessary to reach such a situation, which arises "naturally" here because of the structure of the Galilean group. This makes the phase dynamics second order in time, and one could expect oscillatory behavior.

In the long-wavelength limit, we see from Eq. (6) that $\rho = \mathrm{Re}\, a$ is damped. The main assumption of phase dynamics is that the fast variables, here ρ, follow adiabatically the phase modes ϕ and ψ, and their spatial derivatives. In other words, the phase modes and their spatial derivatives contain all the information about the asymptotic time dependence of a and b. We express this assumption in the *Ansatz*

$$\begin{pmatrix} a \\ b \end{pmatrix} = V[\phi, \psi, \phi_x, \psi_x, \phi_{xx}, \psi_{xx}, \ldots] \qquad (8)$$

and look for an expression for ϕ and ψ in the form[11]

$$\phi_t = \psi, \quad \psi_t = f(\phi, \psi, \partial_x). \qquad (9)$$

We next assume that V and f can be expanded in multiple Taylor series in ϕ, ψ, and ∂_x, and find the expression of V and f at each order using a nonlinear perturbation method similar to the one used for ordinary differential equations.[11] We find up to the fourth order in ∂_x and second order in ϕ and ψ

$$\begin{aligned} &\phi_t = \psi, \\ &\psi_t = \alpha\phi_{xx} = \beta\psi_{xx} + \gamma\phi_{xxxx} + \delta\psi_{xxxx} \\ &\qquad + g\phi_x\phi_{xx} + g'(\phi_x\psi_x)_x, \end{aligned} \qquad (10)$$

where α, β, γ, δ, g, and g' are functions of q that depend on q_0, λ, and σ. We first need the values of α and β to study the linear stability of a perfectly periodic pattern with respect to long-wavelength perturbations, i.e., ϕ and ψ varying like $e^{\eta t + ikx}$ with $k \to 0$. We have

$$\alpha = 2\sigma q(q_0 + q),$$

$$\beta = \lambda + [Q^2 - \sigma q q_0 - (2+\sigma)q^2]/Q^2.$$

We find from (10) the dispersion relation

$$\eta^2 + \beta\eta k^2 + \alpha k^2 = 0.$$

In the limit of small q and for $\sigma q > 0$, we get a pair of damped propagative modes with a propagation speed $(2\sigma q_0 q)^{1/2}$. A stationary instability exists for $\sigma q \times (q_0 + q) < 0$, and a Hopf bifurcation occurs for

$$q = \frac{-\sigma_0 + [(\sigma q_0)^2 + 4(\lambda+1)(\sigma+\lambda+3)]^{1/2}}{2(\sigma+\lambda+3)}, \quad \sigma q(q_0 + q) > 0.$$

These instabilities are suppressed if $\lambda \to \infty$, and are therefore related to the mean drift effect, and thus to *Galilean invariance.* In this limit $\lambda \to \infty$ the second-order derivative in time disappears from Eq. (10) which leads to[12]

$$\phi_{xxt} = \frac{1-3q^2}{1-q^2}\phi_{xxxx} + \ldots .$$

The Eckhaus instability[13] is thus recovered for $q^2 = \frac{1}{3}$ in this singular limit.

A few remarks should be given regarding Eq. (10), in order to stress that its form mainly arises from symmetry constraints. We first observe that it is invariant under the transformations

$$x \to -x, \quad \phi \to -\phi, \quad \psi \to -\psi.$$

This reflects space-reflection symmetry. The invariance of (10) under the transformation

$$\phi \to \phi + c$$

is nothing else than translational invariance. Similarly, the invariance under

$$\psi \to \psi + c, \quad \phi \to \phi + ct$$

reflects Galilean invariance. Consequently, we wish to stress that Eq. (10) is not restricted to our model, but applies for generic one-dimensional patterns with mean flow effect and the symmetry properties listed above.[14]

Finally, we note that $\phi = hx$ (where h is an arbitrary constant) is a particular solution of Eq. (10). This reflects the existence of a band of allowed wave vectors about q_0, for the pattern described by the Eqs. (2) and (3). We consider

$$\phi(x,t) = hx + \epsilon e^{\eta t + ikx}$$

and linearize Eq. (10) in ϵ. We get

$$\eta^2 + (\beta + g'h)k^2\eta + (\alpha + gh)k^2 = 0. \qquad (11)$$

We notice that for h small, $\phi = hx$ represents a periodic pattern of wave number $q + h/Q$; its linear stability in the long-wavelength limit is described by the dispersion relation

$$\eta^2 + \beta(q + h/Q)k^2\eta + \alpha(q + h/Q)k^2 = 0. \qquad (12)$$

Using Eqs. (11) and (12) we get

$$g = \frac{1}{Q}\frac{\partial\alpha}{\partial q}, \quad g' = \frac{1}{Q}\frac{\partial\beta}{\partial q}.$$

At leading order in q, these expressions are in agreement with our computation of the coefficients of Eq. (10), and as noted in Ref. 2, imply that the nonlinearities in phase equations are associated with local changes in the phase propagation speed and in the phase diffusion coefficient. This is a well-known mechanism in nonlinear wave theory.[15]

We now come back to the specific example of thermal convection. In that case the incompressibility condition in the Boussinesq approximation does not allow a slow variation of B in the x direction only, and the coupling between A and B requires a y dependence of these fields. Such long-wavelength disturbances along the axis of the rolls lead to the oscillatory instability,[9] and are described by slow variations of ϕ and ψ on y. With a similar analysis as the one used above we get

$$\phi_t = \psi, \tag{13a}$$

$$\psi_t = \alpha\phi_{yy} + \beta\psi_{yy} + \gamma\phi_{yyyy} + \delta\psi_{yyyy} + g_1\phi_y^2\phi_{yy} + g_2(\phi_y^2\psi_y)_y. \tag{13b}$$

The linear stability analysis is the same as for Eq. (10), and the oscillatory instability corresponds to the Hopf bifurcation. The nonlinear terms of Eq. (13) saturate the oscillatory instability if g_1 and g_2 are positive since they simply renormalize the propagation speed α and the diffusion coefficient β.

In most experimental realizations of thermal convection, the top and bottom boundaries are rigid. This externally breaks Galilean invariance, and thus ψ is damped. However, this damping decreases linearly with the fluid Prandtl number, and can be modeled, for small Prandtl number, by the addition of the term $-\nu\psi$ on the right-hand side of Eq. (13b). The oscillatory instability occurs from the interaction between the neutral translational mode and the slightly damped Galilean mode. Its frequency at onset is finite and increases with the Prandtl number in agreement with the analysis of Clever and Busse.[16] A detailed study of Eqs. (13) and a comparison with the numerical results on the Oberbeck-Boussinesq equations will be given elsewhere.

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[(a)]Also at Observatoire de Nice, B. P. 139, 06003 Nice, France.

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$$\gamma = -\lambda\left(\frac{1-3q^2}{1-q^2}\right) + \frac{\sigma q(q_0+q)[1+q^2+\sigma q(q+q_0)]}{(1-q^2)^2},$$

and δ contributes only at a higher order in space derivatives.

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