Lattice Calculation of Weak Matrix Elements

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We present the first results from a small-lattice ($6^3\times 10$) calculation of nonleptonic weak matrix elements. The $\Delta I=\frac{1}{2}$ rule is studied as a test case. For a lattice meson of approximately the kaon mass we find a significantly enhanced $\Delta I=\frac{1}{2}$ amplitude and a $\Delta I=\frac{3}{2}$ amplitude compatible with zero within our statistics. The dominance of the $\Delta I=\frac{1}{2}$ amplitude appears to be due to a class of graphs called the eye graphs. Qualitatively similar results are found whether or not the charm quark is integrated out *ab initio*. We also report preliminary results on other weak matrix elements.

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A long-standing problem in low-energy hadronic physics has been the calculation of nonleptonic weak matrix elements. Prominent examples are the $\Delta I=\frac{1}{2}$ rule, whose origin has remained obscure, and the *CP*-nonconservation parameters ϵ and ϵ' in neutral-kaon decays. In fact, accurate calculations of the relevant hadronic matrix elements, coupled with existing experimental measurements of ϵ and ϵ', could provide nontrivial tests of the standard model. Lattice Monte Carlo (MC) techniques offer a unique opportunity for the performance of such calculations directly from the fundamental theory. However, the efforts of the past few years suggest that practical difficulties often seriously limit the accuracies attainable with these methods. It thus seems reasonable to begin by studying effects for which even qualitative results can be physically significant.[1] The $\Delta I=\frac{1}{2}$ rule, i.e., the empirical statement that $\Delta I=\frac{1}{2}$ amplitudes are enhanced over the $\Delta I=\frac{3}{2}$ amplitudes by a factor of ~ 20, is one such effect. Because the enhancement is so large, an inaccuracy of order 50% (which is not unusual in current MC calculations) need not mask its origin. The same machinery can, of course, also be used to compute other nonleptonic weak-interaction matrix elements. Here we report the first results from our lattice calculation.

To relate matrix elements amenable to a MC calculation to the experimentally measured $K\to\pi\pi$ amplitudes, three key theoretical ingredients are used: the operator-product expansion and renormalization group (OPE/RG), lattice weak-coupling perturbation theory, and chiral perturbation theory[2] (CPTh). The OPE/RG is required because the characteristic scale for weak interactions is the mass $M_W\sim 80$ GeV of the W boson, while the lattice ultraviolet cutoff (dictated by low-energy hadronic physics and computer time) is $\pi/a\sim 3$ GeV in our calculation (a is the lattice spacing). Thus the W field has to be integrated out from the weak Hamiltonian. When performed in the continuum this procedure leads to two four-quark operators ($O_\pm$) in which the charm quark appears as an explicit field.[3] One can also go further by integrating out the charm quark (the penguin approach).[4-6] One then has six four-quark operators Q_1–Q_6: Q_1–Q_4 are left-left (LL) operators; Q_5, Q_6 are left-right (LR) operators. Since the charm mass is not very large the reliability of this approximation is uncertain, but we study it anyway for purposes of comparison and because it may be useful for coarse-grained lattices. Note that for matrix elements relevant to *CP* nonconservation in the standard model the top quark, at least, must be integrated out by a penguin-type approach (since $m_t >> a^{-1}$).

In order to use the standard results for the OPE/RG from the continuum literature, a lattice weak-coupling calculation, relating matrix elements of continuum operators to their lattice counterparts, is required. Such a relation has the following generic appearance[7] (sum on j, $j\neq i$):

$$O_i^{\rm cont}=[1+(g^2/16\pi^2)Z_{1i}(r,\mu a)]O_i^{\rm latt}+(g^2/16\pi^2)Z_{2ij}(r,\mu a)O_j^{\rm latt}$$
$$+(g^2/16\pi^2)Z_3(r,\mu a)\bar{s}\gamma_\mu(1-\gamma_5)t^a d[\bar{u}\gamma_\mu t^a u+\bar{d}\gamma_\mu t^a d+\bar{s}\gamma_\mu t^a s]. \quad (1)$$

Here the Z's are finite renormalization constants, r is the Wilson parameter, and μ is the continuum renormalization point. The $O(g^2)$ terms can be very important even for weak coupling because naive $O(g^0)$ LL operators can mix at this order with LR operators whose matrix elements are often much larger. Note, however, that the calculations to $O(g^2)$ should be sufficient unless new operators which appear only at $O(g^4)$[8] (such as $\bar{s}\sigma^{\mu\nu}F_{\mu\nu}d$) turn out to have anomalously large matrix elements.

We thus wish to calculate matrix elements of the type $\langle\pi\pi|\bar{s}\Gamma_1 u\bar{u}\Gamma_2 d|K\rangle$, where Γ_i represents an arbitrary Dirac matrix. This is a four-point function and practical considerations make it difficult to evaluate on the lattice. An approximation scheme for light meson

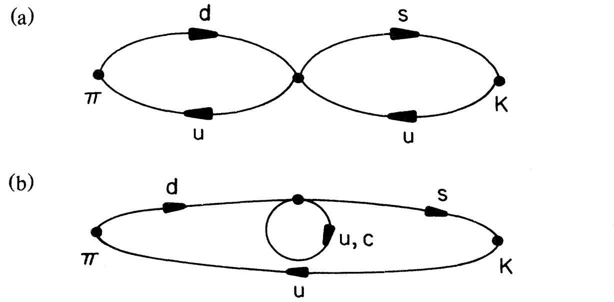

FIG. 1. (a) The "figure-eight" graph; (b) the "eye" graph.

masses, namely, CPTh, allows one to reduce one of the pion fields and relate the $K \to \pi\pi$ matrix element to a suitable linear combination of $K \to \pi$ and $K \to 0$ (0 is the vacuum) matrix elements.[2]

After replacement of the remaining mesons by their interpolating fields, Green's functions such as

$$\langle 0|\bar{d}\gamma_5 u(x)\bar{s}\Gamma_1 u\bar{u}\Gamma_2 d(0)\bar{u}\gamma_5 s(y)|0\rangle$$

result. Wick contraction yields Fig. 1(a) (the "figure-eight" graph)[9] and Fig. 1(b) (the "eye" graph). Despite its quark loop the eye graph should not be eliminated in the quenched approximation since it originates simply from a W-boson correction to a single valence-quark line (see Ref. 1). Furthermore, since it includes all possible gluon corrections, the eye graph cannot be eliminated by normal ordering of the operator.

The inclusion of the eye graphs introduces a technical complication[1] which requires the "exponentiation" of the kaon (i.e., putting a kaon source term into the action).[10] This is equivalent to use of a modified source term in the Gauss-Seidel algorithm. The eye graphs then become tractable to usual (quenched) MC, requiring only about twice the time of a hadron mass calculation.

The background gauge fields were generated by the standard Metropolis MC technique with twelve hits per site on a lattice of size $6^3 \times 10$ with periodic boundary conditions at $\beta = 6/g^2 = 5.7$. Two independent sets of eight configurations were used which evolved to equilibrium from different starting points. One thousand passes were used for thermalization; 500 passes separated each configuration used. Each of the configurations was copied in the time direction to form the background in which $6^3 \times 17$ quark lattices were embedded. The quarks had periodic boundary conditions in the spatial direction and "free"[11] (Neumann) boundary conditions in the time direction (see Ref. 12 for details). By varying the fermion lattice size, we checked that the time boundary effects were no more than a few percent of the final results.

The quark propagators (with Wilson $r = 1$) were calculated for seven values of the hopping constant, $k = 0.094$, 0.123, 0.150, 0.155, 0.162, 0.164, and 0.165; k_c was found to be 0.171 ± 0.002.[13] Table I shows the pseudoscalar meson mass (m_M) and the decay constant f_M (normalized to $f_\pi = 132$ MeV) for $k \geq 0.150$. With $a^{-1} \simeq 1$ GeV,[14] the meson with $k = 0.162$, 0.164, or 0.165 has roughly the mass of a kaon, but note that the lattice meson is made of degenerate quarks, unlike the physical kaon. Note that af_M and (for $k > 0.155$) am_M^2/m_q are fairly constant, indicating compatibility with chiral behavior. The decay constant is found by calculation of the matrix element of the axial-vector current,[15] i.e., from $\langle 0|A_\mu(x)|M(p)\rangle = if_M e^{ip\cdot x}p_\mu$. We use $am_q = \ln[1 + 0.5(k^{-1} - k_c^{-1})]$ to find the quark masses.

Figure 2 shows the $\Delta I = \frac{1}{2}$ and the $\Delta I = \frac{3}{2}$ amplitudes for $K \to \pi$ as a function of k both with and without the penguin approach. When the charm quark was included as an explicit field we used $k = 0.094$; the corresponding $c\bar{c}$ pseudoscalar (i.e., η_c) has $am_M \simeq 3$. As the meson mass gets lighter, the $\frac{3}{2}$ amplitude decreases and for $k = 0.162$–0.165, i.e., $m_M \sim m_K$, it is compatible with zero within statistics. On the other hand, the $\frac{1}{2}$ amplitude is seen to increase for lighter mesons. For $m_M \sim m_K$, the $\frac{1}{2}$ amplitude is much larger than the $\frac{3}{2}$ amplitude. It is interesting that the results from both approaches are in rough agreement.[16]

Table I also shows a ratio of the contribution of the eye graph to that of the figure-eight graph for the K-π

TABLE I. Numerical results for the five lightest meson masses ($\beta = 5.7$, $6^3 \times 17$ fermion lattice, $r = 1$, quenched). Only statistical errors are shown.

k	am_q	am_M	af_M	am_M^2/m_q	aA^{latt} $(K_S \to \pi\pi)$	Eyes/Eights $\langle O_+\rangle$	$\langle O_-\rangle$	$\langle Q_6\rangle$	ξ^{latt}	a^4M_{LL}	a^4M_{LR}
0.150	~ 0.34	1.09 ± 0.05	0.22 ± 0.01	~ 3.4	0.9 ± 0.1	-2	-2	-24	5.7 ± 0.3	-0.109 ± 0.006	0.60 ± 0.12
0.155	~ 0.26	0.90 ± 0.07	0.19 ± 0.01	~ 3.1	1.7 ± 0.2	-3	-5	-26	4.4 ± 0.1	-0.056 ± 0.013	0.52 ± 0.19
0.162	~ 0.15	0.64 ± 0.11	0.15 ± 0.02	~ 2.7	5.6 ± 0.2	$\gtrsim 20$	20	-34	3.7 ± 0.5	$-0.033^{+0.056}_{-0.011}$	$0.38^{+0.21}_{-0.09}$
0.164	~ 0.12	0.57 ± 0.12	0.14 ± 0.04	~ 2.8	9.5 ± 1.9	23	10	-37	3.6 ± 0.6	$0.006^{+0.078}_{-0.016}$	$0.41^{+0.19}_{-0.11}$
0.165	~ 0.10	0.54 ± 0.11	0.14 ± 0.05	~ 2.9	13.2 ± 5.6	18	7	-40	3.5 ± 0.5	$0.007^{+0.094}_{-0.016}$	$0.45^{+0.19}_{-0.13}$

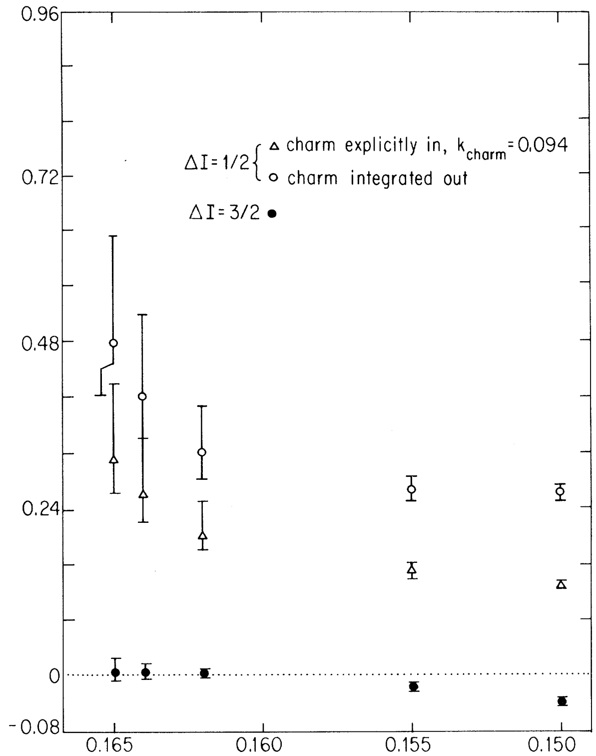

FIG. 2. The $K \to \pi$ amplitudes vs hopping constant k. The $\Delta I = \frac{3}{2}$ amplitude is essentially unchanged whether or not charm is integrated out.

matrix elements of some four-quark operators. The ratio eyes/eights becomes large as the meson mass gets lighter. Since the eye graph is purely $\Delta I = \frac{1}{2}$ while the figure-eight graph is a mixture of $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$, the $\Delta I = \frac{1}{2}$ rule seems to be due to the dominance of eye graphs over figure-eight graphs. We note that the eye-graph contributions in Table I are the result of large cancellations ($\simeq 80\%$) between the up- and the charm-quark loops; retaining either loop alone would give a very misleading result (at least at our renormalization point).

CPTh would also require the $K \to 0$ amplitude (as well as $K \to \pi$) in order to predict the $K \to \pi\pi$ amplitude.[2] Our calculations of $K \to 0$ show that the contribution of this amplitude is $\lesssim 15\%$ of that of $K \to \pi$ and thus can be ignored at this qualitative stage of the project.

There is a feature of Fig. 2 that is disturbing. From CPTh one expects both amplitudes to vanish as m_M goes to zero. The $\Delta I = \frac{1}{2}$ amplitudes in the figure show no such behavior. It is possible that this is a lattice artifact, due to bad chiral properties of Wilson fermions at the not-very-weak coupling of $\beta = 5.7$. However, the fact that both Glashow-Iliopoulos-Maiani subtracted and nonsubtracted operators have similar mass dependence may be an indication that we are seeing the continuum physics of this rather heavy mass range: There is no sign in the eye graphs of spurious, mass-independent contributions (such as occur for $\langle \bar{\psi}\psi \rangle$). Further, our calculations of $K \to 0$ amplitude indicates that non-chiral-invariant operators, e.g., $(r/a)\bar{s}\gamma_5 d$, generated by the eye graphs do not dominate the $\Delta I = \frac{1}{2}$ amplitude.

Since CPTh behavior is not observed, the $K \to \pi\pi$ rate cannot be reliably computed from our lattice $K \to \pi$ amplitude. However, for what it is worth, we show in Table I the calculated (explicit charm) $\Delta I = \frac{1}{2}$ $K \to \pi\pi$ reduced amplitude,[12,17] A^{latt}, which should be compared with the experimental value, $A^{\text{expt}} = 2.25$ GeV. Although the error bars are large, the lack of CPTh behavior is again obvious (in CPTh, A is independent of meson mass), and the magnitude of A^{latt} is quite large for $k = 0.162$ to 0.165. Clearly, better control of the chiral behavior is needed: Future projects will use lower masses, weaker coupling, and various values of Wilson r, including $r = 0$ (Kogut-Susskind fermions).

We have also looked at $\langle \pi | Q_{5,6} | K \rangle$. With the penguin approach and CPTh, this matrix element can be related[18] to ϵ'/ϵ. Thus $|\epsilon'/\epsilon|^{\text{latt}} \simeq 15.6 |\xi^{\text{latt}} s_2 s_3 s_\delta c_2|$, where $s_2 s_3 s_\delta c_2$ are Kobayashi-Maskawa angles and

$$\xi^{\text{latt}} = \frac{\langle \pi^+ | \text{Im}(\tilde{C}_5 Q_5 + \tilde{C}_6 Q_6) | K^+ \rangle}{\langle \pi^+ | C_+ O_+^{\Delta I = 1/2} + C_- O_- | K^+ \rangle} \tag{2}$$

with the C's the Wilson coefficients.[5] From Table I we see that for $k \geqslant 0.162$, ξ^{latt} is fairly independent of k and $\simeq 3.5$. Thus this ratio appears to scale as CPTh suggests and is larger than estimates[18] based on the bag model. Further, we find very good agreement (within $\sim$ 2%) between ξ^{latt} and the corresponding quantity calculated with charm included explicitly. However, since the individual amplitudes do not show CPTh behavior, the relation of ϵ'/ϵ to ξ^{latt} is at this point untrustworthy. We must also emphasize that the matrix elements for the non-GIM-subtracted operators which enter here have a systematic uncertainty originating from possible mixing[19] at higher orders in g^2 with $(r/a)\bar{s}\sigma_{\mu\nu}F^{\mu\nu}d$. Note that even for the left-right operators $Q_{5,6}$, the eye graphs completely dominate over the figure-eight graphs (Table I).

Finally, we have examined the K^0-$\bar{K}^0$ matrix elements of the $\Delta S = 2$ LL and LR operators,[20] i.e., $M_{LL,LR} = \langle K^0 | \bar{s}\gamma_\mu(1-\gamma_5) d\, \bar{s}\gamma^\mu(1 \pm \gamma_5) d | \bar{K}^0 \rangle$. For $am_M \gtrsim 0.9$, the values of M_{LL} are quite consistent with vacuum saturation (computed by use of the lattice values of f_M); however, for $m_M \simeq m_K$, our values for M_{LL} are at this point compatible with zero with very large statistical errors. In contrast, the LR matrix element, M_{LR}, seems to show little variation with meson mass. We find that M_{LR} is two to five times more than implied by vacuum saturation.

To summarize, this work illustrates that lattice MC techniques can be very useful for attacking the old but very difficult problem of the nonleptonic decays of strange particles. Even at this exploratory stage, we find a useful qualitative understanding of the origin of the $\Delta I = \frac{1}{2}$ rule. The numerical values of some other quantities such as M_{LL} and $\xi^{\rm latt}$ appear interesting but more definite statements must await better control of systematics such as mass dependence (CPTh). The theoretical machinery that we have set up will now be used on a bigger (i.e., $12^3 \times 16$ gauge) lattice requiring > 2000 Cray X-MP hours. The goal is a level of accuracy of $\sim 30\%$ in our calculations.

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[7]The constants Z_{1i} and Z_{2ij}, which are the only nonzero constants for four-quark operators with all flavors distinct, have been calculated by G. Martinelli [Phys. Lett. **141B**, 395 (1984)]. We have repeated this calculation and agree with his numerical values for Z_{1i} and Z_{2ij}, but disagree by a few signs and factors of 2 in the definitions of the off-diagonal operators $O_j^{\rm latt}$. We have also calculated the additional graphs necessary when the operators contain quark-antiquark pairs which may contract with each other ("eye" graphs). For $r=1$, we find $Z_3 = \pm [0.062 \pm 0.001 - \frac{8}{3} \ln(a\mu)]$ for $O_\pm$ without Glashow-Iliopoulos-Maiani subtraction. For details see C. Bernard, T. Draper, and A. Soni, to be published.

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[13]One of the sixteen configurations was eliminated because it had an extraordinary low m_M over several time slices and was therefore very sensitive to finite-size limitations. Including it would not change the qualitative conclusions reached here but would increase the statistical errors significantly for $k = 0.162$ and 0.164. See Ref. 12 for details.

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[16]The penguin approach is sensitive to the choice of renormalization point. We choose $\mu = (\Lambda_{\overline{\rm MS}}/\Lambda_{\rm latt})a^{-1}$ (see Ref. 12). Other, perhaps somewhat less reasonable, choices produce variations of $\sim \pm 50\%$. The "explicit charm" method suffers from the fact that our lattice cutoff is not $>> M_{\rm charm}$. Replacing $k_{\rm charm}$ by 0.123 ($am_M \simeq 2$ instead of $\simeq 3$) reduces the $\Delta I = \frac{1}{2}$ amplitude by almost a factor of 2.

[17]The values for $A^{\rm latt}$ in Table I differ from those given in Ref. 12 because of the new method for computing f_M. See Ref. 15.

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[20]Note that only figure-eight graphs contribute to these quantities. Lattice calculations of M_{LL} also appear in Cabibbo, Martinelli, and Petronzio, Ref. 9, and Brower *et al.*, Ref. 9.