Chiral Loops in $\pi^0, \eta^0 \to \gamma\gamma$ and η-η' Mixing

John F. Donoghue, Barry R. Holstein, and Y.-C. R. Lin
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
(Received 23 October 1985)

We compute the one-loop corrections to the Wess-Zumino Lagrangean governing $\pi^0, \eta_8 \to \gamma\gamma$. The SU(3) relation between the two decay rates receives modifications of order $m_K^2 \ln m_K^2$, as does the Gell-Mann–Okubo mass formula. A consistent picture of η-η' mixing emerges from both the two-photon widths and the mass matrix, however with a mixing angle $\theta \approx -20°$ instead of the usual $\theta \approx -10°$.

PACS numbers: 13.40.Hq, 11.40.Ha

Chiral symmetry[1] plays the dominant role in the determination of the masses and decays of the light pseudoscalar mesons $(\pi, \kappa, \eta, \eta')$. If the *u*, *d*, and *s* quarks were all massless the spectrum would consist of a massless octet of Goldstone bosons plus a massive SU(3) singlet (η_0). With nonvanishing quark masses, however, the octet masses become nonzero, with a first-order relation given by the Gell-Mann–Okubo formula

$$m_{\eta_8}^2 = \tfrac{4}{3} m_K^2 - \tfrac{1}{3} m_\pi^2, \qquad (1)$$

where η_8 is the eighth member $(I=0,\ S=0)$ of the octet. However, at this same order in SU(3) breaking, the singlet η_0 will in general mix with η_8, producing the physical eigenstates η, η' given by

$$\eta = \cos\theta\, \eta_8 - \sin\theta\, \eta_0, \quad \eta' = \sin\theta\, \eta_8 + \cos\theta\, \eta_0. \qquad (2)$$

(We assume here that the mixing parameter is energy independent, so that the mixing is orthogonal.) At the same time, chiral symmetry predicts the absolute magnitude of $\pi^0 \to \gamma\gamma$ in a famous relation involving the Adler-Bell-Jackiw triangle anomaly.[2] A corresponding prediction for $\eta_8 \to \gamma\gamma$ can be obtained through use of SU(3). However, the amplitude for the physical process $\eta \to \gamma\gamma$ will be affected by both (i) η-η' mixing and, *at the same order,* (ii) SU(3) breaking of the prediction for $\eta_8 \to \gamma\gamma$ (which has not yet to our knowledge been calculated).

The above description is almost standard textbook material. The conventional estimate of η-η' mixing involves the mass matrix (written in the η_8, η_0 basis)

$$m^2 = \begin{pmatrix} m_{\eta_8}^2 & a \\ a & m_{\eta_0}^2 \end{pmatrix}, \qquad (3)$$

with $m_{\eta_8}^2$ given by Eq. (1), and a and $m_{\eta_0}^2$ unknown. (The use of mass squared is also required by chiral symmetry.) Fitting these parameters with the two known masses, yields a well-known prediction for the mixing angle:

$$\theta \approx -9°. \qquad (4)$$

Recently, however, precise experiments[3] on the two-photon decays of π^0, η, and η' have suggested a mixing angle about *twice* this value, and the consistency of this picture would appear to be seriously threatened. Below we will (i) argue that the experimental analysis is not treating SU(3) breaking consistently unless first-order breaking of $\eta_8 \to \gamma\gamma$ is included, (ii) calculate this correction at one-loop order in chiral perturbation theory using the Wess-Zumino Lagrangean, (iii) calculate the one-loop corrections to the mass-matrix analysis, and (iv) show that consistency is obtained with an angle $\theta \approx -20°$.

First we parametrize the decay amplitudes for $P_i \to \gamma\gamma$, with $P_i = \{\pi^0, \eta_8, \eta_0\}$, as

$$\mathscr{A}(P_i \to \gamma\gamma) = \frac{N_c \alpha}{6\pi \bar{F}_i} C_i \epsilon_{\mu\nu\alpha\beta} \epsilon^{*\mu} q^\nu \epsilon'^{*\alpha} q'^\beta, \qquad (5)$$

where $C_i = \{1, 1/\sqrt{3}, 2(\tfrac{2}{3})^{1/2}\}$, $N_c = 3$ is the number of colors, and $\bar{F}_i = \{\bar{F}_\pi, \bar{F}_8, \bar{F}_0\}$ are not at this stage related to the axial-vector decay constraints. However, use of the axial-vector anomaly plus pion PCAC (partial conservation of axial-vector current) predicts[2] $\bar{F}_\pi = F_\pi = 94$ MeV, a value which is in good accord with experiment. If one then includes η_8-η_0 mixing one finds for the η, η' decay rates

$$\frac{\Gamma(\eta \to \gamma\gamma)}{\Gamma(\pi \to \gamma\gamma)} = \frac{1}{3} \frac{m_\eta^3}{m_\pi^3} \left(\frac{F_\pi \cos\theta}{\bar{F}_8} - \frac{F_\pi \sin\theta}{\bar{F}_0} \right)^2,$$
$$\frac{\Gamma(\eta' \to \gamma\gamma)}{\Gamma(\pi \to \gamma\gamma)} = \frac{8}{3} \frac{m_{\eta'}^3}{m_\pi^3} \left(\frac{F_\pi \sin\theta}{\bar{F}_8} + \frac{F_\pi \cos\theta}{\bar{F}_0} \right)^2. \qquad (6)$$

Use of exact SU(3) would predict $\bar{F}_8 = F_\pi$, and one could in principle obtain a fit to the two ratios, using $\bar{F}_0$ and θ as unknowns. But however practical this approach may be, it is not consistent in that to first order in SU(3) breaking one must allow both $\bar{F}_8 \neq F_\pi$ and $\theta \neq 0$. While we will argue below that the effect of the mixing angle is much larger than SU(3) breaking in $\bar{F}_8$, this is not known *a priori.*

The predictions of chiral symmetry for physics related to the anomaly are contained in the effective Lagrangean first given by Wess and Zumino.[4] With the inclusion of electromagnetism, it has the form

$$\Gamma_{\mathrm{WZ}}=N_c\Gamma(U)-N_c e\int d^4x\,A_\mu J^\mu+(i\alpha/6\pi)N_c\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu A_\alpha \mathrm{Tr}[Q^2(\partial_\beta U)U^{-1}+Q^2U^{-1}(\partial\beta U)+\tfrac{1}{2}QUQU^{-1}(\partial_\beta U)U^{-1}+\tfrac{1}{2}QU^{-1}QU(\partial_\beta U^{-1})U], \quad (7)$$

where

$$Q=\begin{pmatrix}\frac{2}{3} & 0 & 0\\ 0 & -\frac{1}{3} & 0\\ 0 & 0 & -\frac{1}{3}\end{pmatrix} \quad (8)$$

is the charge matrix,

$$U=\exp i\lambda^A\phi^A/F, \quad (9)$$

with ϕ^A $(A=1,\ldots,8)$ being the octet of pseudoscalar fields, and

$$J_\mu=(1/48\pi^2)\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[Q(\partial_\nu UU^{-1})(\partial_\alpha UU^{-1})(\partial_\beta UU^{-1})+Q(U^{-1}\partial_\nu U)(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)]. \quad (10)$$

At tree level $F=F_\pi$, and the two-photon term in Eq. (7) gives the usual chiral predictions for $\pi^0,\eta_8\to\gamma\gamma$ in the SU(3) limit.

Loop corrections at low energies in chiral theories are, in fact, well defined despite the fact that the theory is nonrenormalizable.[1] Terms with the nonanalytic structure $m^2\ln m^2$ are the leading corrections at one-loop order. Subleading pieces of order m^2 are not unambiguous predictions because their effect is the same as higher-derivative tree-level Lagrangeans whose coefficients are in general not known. The decay-rate ratio calculated below is in fact free of these m^2 terms when expressed in terms of the physical values of F_π and F_η; however, the mass shift could in principle depend on them. Nevertheless, as is standard, we will limit our calculation to model-independent corrections of order $m_K^2\ln m_K^2$.

The one-loop renormalization involves the diagrams of Figs. 1(a) and 1(b) as well as wave function renormalization and the one-loop renormalization of the axial-vector decay constants F_π and F_{η_8}. For the latter we find from the diagram in Fig. 2

$$F_\pi=F\left[1+\frac{1}{2}\frac{m_K^2}{(4\pi F_\pi)^2}\ln\frac{m_K^2}{\mu^2}\right],$$
$$F_{\eta_8}=F\left[1+\frac{3}{2}\frac{m_K^2}{(4\pi F_\pi)^2}\ln\frac{m_K^2}{\mu^2}\right], \quad (11)$$

where μ represents the renormalization point. Displaying the results of the one-loop calculation to $P\to\gamma\gamma$ in the order given above, we find then

$$\mathscr{A}(\pi^0\to\gamma\gamma)=\frac{N_c\alpha}{6\pi F_\pi}\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu q_\nu\epsilon'_\alpha q'_\beta\left[1+\left(\frac{4}{3}-2+\frac{1}{6}+\frac{1}{2}\right)\frac{m_K^2}{(4\pi F_\pi)^2}\ln\frac{m_K^2}{\mu^2}\right]=\frac{N_c\alpha}{6\pi F_\pi}\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu q_\nu\epsilon'_\alpha q'_\beta,$$
$$\mathscr{A}(\eta_8\to\gamma\gamma)=\frac{N_c\alpha}{6\sqrt{3}\pi F_{\eta_8}}\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu q_\nu\epsilon'_\alpha q'_\beta\left[1+\left(1-2-\frac{1}{2}+\frac{3}{2}\right)\frac{m_K^2}{(4\pi F_\pi)^2}\ln\frac{m_K^2}{\mu^2}\right]=\frac{N_c\alpha}{6\sqrt{3}\pi F_{\eta_8}}\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu q_\nu\epsilon'_\alpha q'_\beta, \quad (12)$$

where we have displayed the result using the physical F_π and F_η instead of the "bare" parameter F and have neglected terms of order $m_\pi^2\ln m_\pi^2$. As expected, the pion amplitude is unrenormalized at one-loop order, aside from the implicit renormalization of F_π. The result for the η_8 is particularly simple and pleasing, $F_8=F_{\eta_8}$. However, this *does* imply some SU(3) breaking as $F_{\eta_8}\neq F_\pi$. In fact the one-loop prediction is

$$\frac{F_{\eta_8}}{F_\pi}=1-\frac{m_K^2}{(4\pi F_\pi)^2}\ln\frac{m_K^2}{\mu^2}\approx 1.25 \quad (13)$$

for $\mu\approx 1$ GeV. The renormalization of the η_8 mass[5] can be performed in a similar manner by use of the self-energy diagrams in Fig. 3. Expressed in terms of the renormalized pion and kaon masses we find

$$m_{\eta_8}^2=\frac{4}{3}m_K^2-\frac{1}{3}m_\pi^2-\frac{2}{3}\frac{m_K^2}{(4\pi F_\pi)^2}\ln\frac{m_K^2}{\mu^2}$$
$$\approx(0.61\ \mathrm{GeV})^2. \quad (14)$$

We can now address the phenomenology associated

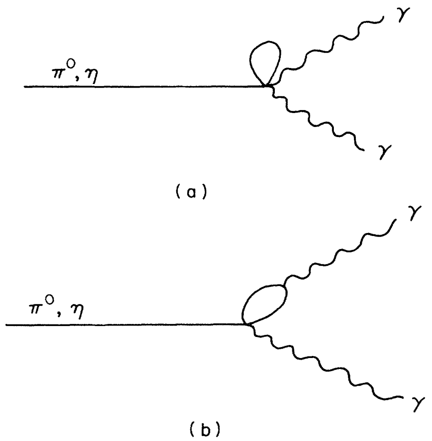

FIG. 1. The one-loop diagrams for the renormalization of $\pi^0 \rightarrow \gamma\gamma$ and $\eta^0 \rightarrow \gamma\gamma$.

with these results. The recent two-photon data,[3]

$$\begin{aligned} &\Gamma(\pi^0 \rightarrow \gamma\gamma) = 7.3 \pm 0.2 \text{ eV}, \\ &\Gamma(\eta \rightarrow \gamma\gamma) = 0.56 \pm 0.04 \text{ keV}, \\ &\Gamma(\eta' \rightarrow \gamma\gamma) = 4.16 \pm 0.30 \text{ keV}, \end{aligned} \tag{15}$$

yield reduced ratios

$$\rho_\eta \equiv \frac{3m_\pi^2}{m_\eta^3} \frac{\Gamma(\eta \rightarrow \gamma\gamma)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = 3.4 \pm 0.3 = \left[\frac{F_\pi \cos\theta}{\bar{F}_8} - \sqrt{8}\frac{F_\pi}{\bar{F}_0}\sin\theta\right]^2 \tag{16}$$

and

$$\rho_{\eta'} \equiv \frac{3}{8} \frac{m_\pi^3}{m_{\eta'}^3} \frac{\Gamma(\eta' \rightarrow \gamma\gamma)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = 0.60 \pm 0.05 = \left[\frac{F_\pi}{\bar{F}_0}\cos\theta + \frac{1}{\sqrt{8}}\frac{F_\pi}{\bar{F}_8}\sin\theta\right]^2. \tag{17}$$

The departure of the first of these numbers from unity is a clear signal for SU(3) breaking and/or η-η' mixing. Using the result of our one-loop calculation ($\bar{F}_8/F_\pi = F_\eta/F_\pi = 1.25$), we find a mixing angle

$$\theta = -(23 \pm 3)^\circ \tag{18}$$

plus

$$\bar{F}_0/F_\pi = 1.04 \pm 0.04. \tag{19}$$

The mixing angle is about 5° larger than would have been obtained if chiral corrections to $\bar{F}_8$ had been neglected.

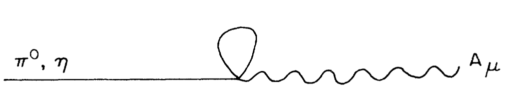

FIG. 2. The one-loop diagram contributing to the renormalization of the pseudoscalar decay constants F_π, F_η.

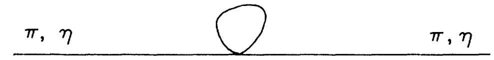

FIG. 3. The one-loop diagram leading to mass renormalization.

The mass-matrix analysis can likewise be done as in Eq. (3), however with use of Eq. (14) for $m_{\eta_8}^2$. Diagonalization to the physical values of the masses requires

$$a = -0.196, \quad m_{\eta_8}^2 = (0.92 \text{ GeV})^2, \tag{20}$$

and predicts

$$\theta = -19.5^\circ. \tag{21}$$

It is amusing to note that the above phenomenological value of a is close to the number predicted[6] in the limit $N_c \rightarrow \infty$,

$$a = -(2\sqrt{2}/3)(m_K^2 - m_\pi^2) = -0.217. \tag{22}$$

However, more important, the mixing angle is much larger than previously obtained in Eq. (4) and is consistent with that emerging from the two-photon analysis.

The use of one-loop corrections in the $m^2 \ln m^2$ limit of chiral SU(2) has proven to be useful in comparison with experiment.[1] The corresponding usage in chiral SU(3) has not been as successful.[7] The reason presumably is that m_K^2 effects, such as can be obtained from higher-derivative chiral Lagrangeans, are not numerically small compared to $m_K^2 \ln m_K^2$. Because of the simple, and the fairly small, size of the correction to the η_8 decay rate, and the fact that there are no remaining m_K^2 effects in the one-loop correction to the decay-rate ratio, one might hope that we have identified the major correction to $\bar{F}_8$. Unless our calculation is far wrong, two-photon rates are a reliable and sensitive measure of the mixing angle. In the mass matrix, however, relatively small modifications could have a sizable effect on the mixing angle. In any case, we feel that our calculation does point to the resolution of the apparent problem with the previous mass-matrix analysis; i.e., a small modification of the Gell-Mann–Okubo mass relation such as expected from loop effects can significantly increase the mixing angle. In addition, it is rather remarkable that the $m^2 \ln m^2$ loop effects yield a very consistent picture of both two-photon decays and mass-matrix mixing.

This research was supported in part by the National Science Foundation.

[1]For reviews see J. Gasser and H. Leutwyler, Phys. Rep. **87C**, 77 (1982); H. Pagels, Phys. Rep. **16C**, 219 (1975).

[2]S. L. Adler, Phys. Rev. **117**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969).

[3]S. Cooper, in Proceedings of the International Europhysics Conference on High-Energy Physics, Bari, Italy, June 1985 (to be published); B. C. Shen, in *Proceedings of the Annual Meeting of the Division of Particles and Fields of the American Physical Society, Santa Fe, New Mexico, 1984,* edited by T. Goldman and M. M. Nieto, (World Scientific, Singapore, 1985), p. 222.

[4]J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[5]This mass shift has been previously calculated by P. Langacker and H. Pagels, Phys. Rev. D **10**, 2904 (1974).

[6]C. B. Thorn, private communication; J. F. Donoghue, in *Experimental Meson Spectroscopy—1983,* edited by S. J. Lindenbaum, AIP Conference Proceedings No. 123 (American Institute of Physics, New York, 1984), p. 107.

[7]J. Bijnens, H. Sonoda, and M. B. Wise, California Institute of Technology Report No. CALT-68-1221 (to be published); J. F. Donoghue and B. R. Holstein, University of Massachusetts Report No. UMHEP-228 (to be published).