Measurement of the Wigner Function

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An operational procedure is given for determining experimentally, in principle at least, the Wigner function for an ensemble of particles. This manner of "measuring" a quantum state, whether pure or mixed, via its Wigner function, seems the simplest possible, and closely parallels the method one might use in classical mechanics to determine a (true) phase-space probability density.

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The Wigner function, although it is not strictly positive, provides a phase-space representation of quantum mechanics which has many desirable features.[1] A point which does not seem to have been much discussed is whether the Wigner function has any operational significance, i.e., whether it can be recorded experimentally.[2] We give here a simple procedure for doing so, which seems, in fact, to be the simplest manner of "measuring" a quantum state; moreover, it closely parallels the method one might use in classical mechanics to determine a (true) phase-space probability density, so that the Wigner function *simulates* a phase-space distribution function not only formally, but operationally also. For simplicity, we consider a single particle of mass m moving in one dimension.

We first explain what we mean by "measuring a quantum state." In classical mechanics, the state of a particle is specified by its position and momentum, and is directly observable; moreover, the act of observation disturbs the state as little as desired (in principle, at least).

A quite different situation prevails in quantum mechanics: Here, all that can be observed in a measurement is into which one of a complete orthonormal set of states $\{\psi_n,\ n=0,1,2,\ldots\}$, determined by the apparatus, the particle "collapses" as a result of the observation itself, which thus uncontrollably disturbs the state of the particle. The probability that the particle collapse into the state ψ_i, or (speaking loosely) that it be "found" in the state ψ_i (which does not mean that it was in that state *before* the measurement), is equal to the *transition probability* $\langle\psi_i|\rho|\psi_i\rangle$, where ρ is the state operator (or density matrix) of the particle *before* the measurement. It is more usual to refer quantum measurements to *observables*[3] (i.e., Hermitian operators whose eigenstates form a complete orthonormal set) thus: If the ψ_n are the eigenstates (eigenvalues a_n) of the Hermitian operator A, and if the particle is "found" in the state ψ_i, then one says that he has measured the observable A and found the value a_i. But as emphasized by Wigner,[4] the measurement basically refers to the set ψ_n.

It follows from the above that the state of an individual particle is *unobservable*, even in principle: While a single measurement cannot, on the one hand, yield sufficient information to reconstruct the particle's state operator, on the other hand it uncontrollably perturbs the state, thereby eliminating the possibility of gathering the remaining information from subsequent measurements. However, the notion of "measuring a state" is meaningful when it refers not to an individual particle, but rather to a specific preparation procedure; i.e., the following question may be posed: Is it possible, by performing a sufficient number of measurements on *different* members of an ensemble of similarly prepared particles, to determine the state operator ρ describing the ensemble?

This question seems to have been seldom addressed in the literature. A way of measuring states has been indicated by Kemble,[5] which, however, applies only to cases where it is known *a priori* that a pure state (always the same) is produced. But most preparation procedures yield mixed states, due, e.g., to unpredictable fluctuations in the state of the apparatus used, and/or to residual quantum correlations of the particle with other objects (e.g., parts of the apparatus). D'Espagnat[6] has remarked that measurement of the set of observables $\pi_{ij\pm}=(\frac{1}{i})(|i\rangle\langle j|\pm|j\rangle\langle i|)$, where $|i\rangle, |j\rangle, \ldots,$ is a complete orthonormal set, would allow deduction of the state operator ρ, whether pure or mixed. Although it is not clear that every "observable" *can* be measured, Lamb[7] has given, for the case of observables depending only on positions and momenta, an explicit prescription for *effectively* measuring them. This, however, requires the use of a large number of different potentials of generally complicated shapes, so that determining ρ by measuring the operators $\pi_{ij\pm}$ in this manner would be extremely laborious. We will see that a simpler approach, requiring only one or two simple potential shapes, consists in determining ρ by measuring its Wigner function.

Let us first imagine the procedure whereby a classical physicist might determine experimentally the probability $f(X,P,T)$ that a "classical" particle (e.g., an electron), prepared according to some specific rules (but which still leave an element of randomness, due, e.g., to thermal fluctuations) be at X with momentum P at time T. We describe two closely related possible methods (our overall approach is very much influenced by Lamb's classic paper[7]).

Method 1.—We put ourselves in a reference frame moving with uniform speed $v=P/m$ relative to the laboratory, and turn on in that frame at time T a potential well $V(x-X)$ whose minimum lies at X [e.g., a harmonic well $\frac{1}{2}m\omega^2(x-X)^2$]. If the particle is then found to lie at rest at the bottom of the well, as evidenced, e.g., by the absence of any emitted radiation, we infer that the particle was at X with momentum P at time T. The relative number of times this happens, in many repetitions of this experiment on different similarly prepared particles, yields the probability $f(X,P,T)$.

Method 2.—An alternative method, avoiding the use of moving apparatus, is to first shift the momentum by $-P$ at time T by applying an impulsive force $-P\delta(t-T)$ derived from a linear potential $xP\times\delta(t-T)$; immediately thereafter, the potential $V(x-X)$ is turned on (in the laboratory frame this time); again, if the particle is found at rest, we infer that its state was (X,P) at time T.

Because of the experimental imprecisions, there is a nonvanishing probability $\chi(\Delta x,\Delta p)$ that even if the particle was at $(X+\Delta x,P+\Delta p)$ at time T, it was nevertheless recorded as (X,P). Thus, what is measured in fact is not $f(X,P)$, but rather the convolution

$$\Gamma(X,P)=\int dx\int dp\,\chi(x-X,p-P)f(x,p), \qquad (1)$$

i.e., the relative probability that the particle be in a "fuzzy" neighborhood of (X,P) [this is, in general, not expected to be normalized, i.e., $\int dX\,dP\,\Gamma=\int dx\,dp\,\chi\neq\int dx\,dp\,f=1$]. Usually, the function $\chi(x,p)$ is unknown, and can only be roughly estimated. Note that if χ were known, then one could deduce (deconvolute) $f(x,p)$: By taking the Fourier transform of (1), we get $\tilde\Gamma(k,s)=\tilde\chi(k,s)\tilde f(k,s)$, where $\tilde\Gamma(k,s)=\int dx\,dp\,e^{ikx-isp}\,\Gamma(x,p)$, etc., and we can solve for $\tilde f(k,s)$ [provided $\tilde\chi(k,s)\neq 0$ almost everywhere].

For the classical physicist, the particle is sharply localized at *some* phase-space point (x,p), so that the "sharp" phase-space probability density $f(x,p)$ ["sharp" in the sense that it is the probability that the particle be at *exactly* the point (x,p), not just in a fuzzy neighborhood of it] is physically real, and could in principle be measured with arbitrary accuracy (if only he had good enough instruments). Not so for a quantum physicist, for the uncertainty principle forbids states which are sharply localized in phase space, so that it is meaningless to speak of the probability that a particle be at exactly the point (x,p); for him, a "sharp" $f(x,p)$ does not exist, and he would give the following analysis of the above experiments.

Method 1.—In the moving frame, the state operator "seen" is[8]

$$\tilde\rho(t)=D_{v(t-T),-P}\,\rho(t)\,D^{-1}_{v(t-T),-P},\quad D_{xp}=e^{i(p\hat x-x\hat p)/\hbar},$$

where ρ is the particle's state operator in the laboratory frame, and D_{xp} effects phase-space translations ($\hat x$ and $\hat p$ are the position and momentum operators). Let ϕ^n and E^n be the eigenstates and energies of the Hamiltonian $H=\hat p^2/2m+V(\hat x)$; then those of $H_{X0}=\hat p^2/2m+V(\hat x-X)$ are ϕ^n_{X0} and E^n, where we set

$$|\phi^n_{xp}\rangle=D_{xp}|\phi^n\rangle,$$

$$H_{xp}=D_{xp}HD_{xp}^{-1}=\frac{(\hat p-p)^2}{2m}+V(\hat x-x).$$

Hence, at times $t>T$, after $V(x-X)$ has been turned on in the moving frame, the state operator in that frame is (dropping $\hbar$'s for simplicity)

$$\tilde\rho(t)=\exp[-i(t-T)H_{X0}]\tilde\rho(T)\exp[i(t-T)H_{X0}]$$
$$=\sum_{m,n}\exp[-i(t-T)(E^n-E^m)]\,|\phi^n_{X0}\rangle\langle\phi^n_{XP}|\rho(T)|\phi^m_{XP}\rangle\langle\phi^m_{X0}|, \qquad (2)$$

where we used $\langle\phi^n_{X0}|\tilde\rho(T)|\phi^m_{X0}\rangle=\langle\phi^n_{XP}|\rho(T)|\phi^m_{XP}\rangle$. What is measured, in the counting of the relative number of times that no emission of radiation is observed, is the (time-independent) transition probability $\langle\phi^0_{X0}|\tilde\rho(t)|\phi^0_{X0}\rangle$ to the ground state ϕ^0_{X0}; according to (2), this is equal to

$$\Gamma_{\phi^0}(X,P,T)=\langle\phi^0_{XP}|\rho(T)|\phi^0_{XP}\rangle. \qquad (3)$$

Method 2.—The state operator $\rho(T+0)$ just after application of the pulse potential $xP\delta(t-T)$ is related to $\rho(T-0)$ just before by

$$\rho(T+0)=D_{0,-P}\,\rho(T-0)\,D^{-1}_{0,-P},$$

since the time evolution operator is

$$\exp\left[-i\int_{T-0}^{T+0}dt[\hat p^2/2m+\delta(t-T)P\hat x]\right]=D_{0,-P}.$$

Then, at times $t>T$, after $V(x-X)$ has been turned on in the laboratory frame, the state operator $\rho(t)$ in that frame is given by (2), but with $\tilde\rho(t)$, $\tilde\rho(T)$, and $\rho(T)$ replaced by $\rho(t)$, $\rho(T+0)$, and $\rho(T-0)$, respectively. What is measured is

$$\langle\phi^0_{X0}|\rho(t)|\phi^0_{X0}\rangle=\Gamma_{\phi^0}(X,P,T-0)$$

again.[9]

The function (3) is the probability of "finding" the particle in the state ϕ^0_{XP}, which is only *partially* localized about (X,P); optimal localization is obtained if the potential $V(x)=\frac{1}{2}m\omega^2x^2$ is harmonic, in which case

$$|\phi^0_{XP}(x)|^2=(2\pi\Delta x^2)^{-1/2}\exp[-\tfrac{1}{2}(x-X)^2/\Delta x^2],$$
$$|\tilde\phi^0_{XP}(p)|^2=(2\pi\Delta p^2)^{-1/2}\exp[-\tfrac{1}{2}(p-P)^2/\Delta p^2], \qquad (4)$$

where $\Delta x=(2m\omega/\hbar)^{-1/2}$ and $\Delta p=(m\hbar\omega/2)^{1/2}$ [the corresponding Γ_{ϕ^0} is then known as the Husimi distribution[10]; the more general distributions, corresponding to arbitrary potentials $V(x)$, were first considered by Klauder[11]]. For the quantum physicist, Γ_{ϕ^0} for V harmonic is the "sharpest" phase-space distribution allowed by the uncertainty principle.

Also, he might propose that a more "complete" quantum experiment would be to measure the whole set of transition probabilities $\langle\phi^n_{XP}|\rho(T)|\phi^n_{XP}\rangle$, $n=0,1,2,\ldots$ (i.e., measure H_{XP}),[12] e.g., by analyzing the radiation emitted after time T, noticing that by (2), the probability that the particle radiate a total energy E^n-E^0 (via one or several photons in cascade), i.e., to be "found" in the state ϕ^n_{X0}, is

$$\langle\phi^n_{X0}|\tilde\rho(t)|\phi^n_{X0}\rangle=\langle\phi^n_{XP}|\rho(T)|\phi^n_{XP}\rangle$$

(because the energy is emitted in discrete amounts, the quantum measurements can be made much more precise than the classical ones which involve continuous energies).

Heeding these suggestions, and then being much impressed by the fact that he indeed observes the discrete energy spectrum predicted by his quantum colleague, the classical physicist might yet argue that even though a particle cannot be prepared or observed, or even described by conventional quantum mechanics, in a state better localized in phase space than (4), it may still have, *in reality,* a sharp position and momentum (here he may seek support in Heisenberg's famous statement "the uncertainty principle does not refer to the past"),[13] so that a "sharp" (in our previous sense) $f(x,p)$ is still a valid concept [of course, this "sharp" $f(x,p)$ cannot itself be sharply localized, of the form $\delta(x-X)\delta(p-P)$, say, since it must reflect the uncertainty relations inherent in the preparation process]. Moreover, considering (4), he might conjecture that the quantum analysis provides a definite expression for $\chi(x,p)$ [see Eq. (1)] when $V(x)=\frac{1}{2}m\omega^2x^2$, to wit,

$$\chi(x,p)=(2\pi\hbar)(4\pi^2\Delta x^2\Delta p^2)^{-1/2}\exp[-\tfrac{1}{2}(x/\Delta x)^2-\tfrac{1}{2}(p/\Delta p)^2] \quad (5)$$

[the factor $2\pi\hbar$ is deduced from $\int dx\,dp\,\chi=\int dx\,dp\,\Gamma_{\phi^0}=2\pi\hbar$] where he made the assumption, quite plausible in view of the Gaussian forms of (4), that the x and p "errors" are uncorrelated; thus, he can now even calculate $f(x,p)$: Using (5) in (1), $\Gamma(x,p)$ having been measured with $V=\frac{1}{2}m\omega^2x^2$, he solves for $f(x,p)$. To his dismay, he finds that $f(x,p)$ is not everywhere positive in general. What he has obtained is, in fact, the Wigner function $F^{\mathrm{W}}(x,p)$, which is *not* a phase-space probability density (but is tantalizingly close to one).

The meaning of $f^{\mathrm{W}}(x,p)$ appears most clearly in the Liouville-space formalism,[14] wherein ordinary quantum operators are viewed as vectors. Introducing a bra-ket notation in that space, we associate with every operator A an L-ket $|A\}$ and an L-bra $\{A|$. The scalar product is defined as $\{A|B\}=(2\pi\hbar)^N\mathrm{Tr}A^\dagger B$, where $N=1$ is the number of degrees of freedom. The L-kets

$$|xp\}=(\pi\hbar)^{-1}|\Pi_{xp}\},\quad \Pi_{xp}=D_{xp}\Pi D_{xp}^{-1},$$

where Π is the parity operator ($\Pi|x\rangle=|-x\rangle$), form a complete orthonormal set:

$$\{x'p'|xp\}=\delta(x'-x)\delta(p'-p),\qquad \int dx\int dp\,|xp\}\{xp|=1_L. \quad (6)$$

The Wigner function is (up to multiplicative constant) the $|xp\}$ representative of the state operator[15] ρ:

$$f^{\mathrm{W}}(x,p)=(2\pi\hbar)^{-1}\{xp|\rho\}=(\pi\hbar)^{-1}\mathrm{Tr}\Pi_{xp}\rho. \quad (7)$$

The connection with the definition given earlier [viz., Eq. (1) with χ given by (5) and Γ by (3) for $V=\frac{1}{2}m\omega^2x^2$] is obtained if we note that

$$\Gamma_{\phi^0}(X,P)=(2\pi\hbar)^{-1}\{|\phi^0_{XP}\rangle\langle\phi^0_{XP}|\rho\}=(2\pi\hbar)^{-1}\int dx\,dp\,\{|\phi^0_{XP}\rangle\langle\phi^0_{XP}||xp\}\{xp|\rho\}$$
$$=2\pi\hbar\int dx\,dp\,f^{\mathrm{W}}_{\phi^0}(x-X,p-P)f^{\mathrm{W}}(x,p), \quad (8)$$

where $f^{\mathrm{W}}_{\phi^0}(x,p)=(2\pi\hbar)^{-1}\{xp||\phi^0\rangle\langle\phi^0|\}$ is the Wigner function for the pure state ϕ^0; we recover (1) by setting $\chi=2\pi\hbar f^{\mathrm{W}}_{\phi^0}$, which equals (5) if ϕ^0 is the ground state of the harmonic oscillator $\hat p^2/2m+\frac{1}{2}m\omega^2\hat x^2$. Equation (8) also shows that we may use an arbitrary potential $V(x)$ in our experiments, and still be able to deconvolute f^{W} from Γ_{ϕ^0} [where ϕ^0 is now the ground state of $\hat p^2/2m+V(\hat x)$], by using $\chi=2\pi\hbar f^{\mathrm{W}}_{\phi^0}$ in (1) [the x and p "errors" are then correlated in general, unlike in (5)].

According to (6) and (7), $f^{\mathrm{W}}(x,p)$ is a sharp phase-space *representation* of ρ, where "sharp" here refers to the δ-function phase-space orthogonality of the L-kets $|xp\}$. However, because Π_{xp} is not a density matrix (not being positive definite), the L-kets $|xp\}$ *are not states.* Herein lies the fundamental difference between $f^{\mathrm{W}}(x,p)$ and the classical idealization that was a sharp phase-space probability density $f^{\mathrm{cl}}(x,p)$: In contrast to the latter, $f^{\mathrm{W}}(x,p)$ is not a density on (or a transition probability to) physical *states*. Physically, $f^{W}(x,p)$ is an *expectation value*, namely that of the parity operation Π_{xp} about the phase-space point (x,p).[15] By contrast, $\Gamma_{\phi^0}(x,p)$ is a "fuzzy" phase-space representation[11] of $|\rho\}$, in terms of the *nonorthogonal* set of *states* $||\phi_{xp}^0\rangle\langle\phi_{xp}^0|\}$, but *is* a transition probability. Thus, $f^{W}(x,p)$ and $\Gamma_{\phi^0}(x,p)$ each share only part of the attributes of $f^{\mathrm{cl}}(x,p)$.

As we saw earlier, it is $\Gamma_{\phi^0}(x,p)$ which is observed "classically," whence $f^{W}(x,p)$ can be deduced via Eq. (8). But $f^{W}(x,p)$ is the expectation value of an *observable*, Π_{xp}, and should, in principle, be directly measurable; this can indeed be done, in effect by the same experiments as already described: The eigenvalues of Π_{xp} are ± 1 (since $\Pi_{xp}^2=1$), and a complete set of eigenstates ϕ_{xp}^n may be gotten by displacing in phase space any complete set of states ϕ^n of definite parity about the origin, i.e.,

$$|\phi_{xp}^n\rangle = D_{xp}|\phi^n\rangle,$$

$$\phi^n(-x) = (-1)^n\phi^n(x) \quad (n=0,1,2,\ldots).$$

Aware that Hamiltonians are, as stressed by Lamb,[7] ideal observables from an operational point of view,[16] let us choose the ϕ^n as the eigenstates of $H = \hat{p}^2/2m + V(\hat{x})$, where $V(x)=V(-x)$ is *any* convenient symmetric potential (e.g., $V=\frac{1}{2}m\omega^2x^2$). Measurement of Π_{XP}, or its expectation value

$$\mathrm{Tr}\Pi_{XP}\rho(T) = \sum_n(-1)^n\langle\phi_{XP}^n|\rho(T)|\phi_{XP}^n\rangle,$$

then amounts to measurement of the set of transition probabilities $\langle\phi_{XP}^n|\rho(T)|\phi_{XP}^n\rangle$ (or H_{XP}); but this is precisely what is done in the "complete" quantum experiments suggested previously.

To summarize, we first devised "classical" experiments to measure a "sharp" phase-space probability density $f^{\mathrm{cl}}(x,p)$, a concept idealized from everyday macroscopic experience. The quantum analysis of these experiments reveals that what is measured is in fact a fuzzy phase-space density $\Gamma_{\phi^0}(x,p)$, subject to the uncertainty principle. But if one surmises that a "sharp" $f(x,p)$ nevertheless exists *in reality*, and tries to calculate it by deconvoluting $\Gamma_{\phi^0}(x,p)$ in a naive manner, one *almost* obtains such an idealized function: One gets the Wigner function $f_\rho^{W}(x,p)$, which is indeed a sharp phase-space representation of ρ, but not a probability density on physical states, and is not always positive. Remarkably, it is possible, by expanding the above "classical" experiments into "complete" quantum experiments, i.e., by measuring a complete set of transition probabilities instead of just a single one [yielding $\Gamma_{\phi^0}(x,p)$], to measure $f_\rho^{W}(x,p)$ directly. This avoids the delicate deconvolution involved above, and seems to be the simplest possible manner of "measuring" a quantum state [measuring $f_\rho^{W}(x,p)$ is equivalent to measuring ρ, since the former is a sharp representation of the latter].

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[2]This question is raised by K. Wodkiewicz [Phys. Rev. Lett. **52**, 1064 (1984)], who seems to imply, however, that the Wigner function is not operational.

[3]See, e.g., P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford Univ. Press, Oxford, 1958) 4th ed.; J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, 1955).

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[6]B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Reading, Massachusetts, 1976), 2nd ed., p. 57.

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[8]See, e.g., W. Pauli, *General Principles of Quantum Mechanics* (Springer, Berlin, 1980), p. 19.

[9]Another manner of measuring $\Gamma_{\phi^0}(x,p)$, via a scattering experiment, is given in Ref. 2. Our approach is closer to Lamb's in Ref. 7.

[10]K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940).

[11]J. R. Klauder, J. Math. Phys. **4**, 1055 (1963).

[12]In the three-dimensional case, one might use $V(x,y,z)=\frac{1}{2}m(\omega_1^2x^2+\omega_2^2y^2+\omega_3^2z^2)$ where ω_1, ω_2, and ω_3 are incommensurate, such as to avoid degeneracies in the set of energies $\hbar(n_1\omega_1+n_2\omega_2+n_3\omega_3)$, $n_i=0,1,2,\ldots$, thus allowing each degree of freedom to be analyzed separately.

[13]W. Heisenberg, *The Physical Principles of the Quantum Theory* (Dover, New York, 1949), p. 20.

[14]See, e.g., U. Fano, Rev. Mod. Phys. **29**, 74 (1957), and in *The Many-Body Problem,* edited by E. R. Caianiello (Academic, New York, 1964).

[15]B. R. Mollow, Phys. Rev. **162**, 1256 (1967); A. Royer, Phys. Rev. **15**, 449 (1977).

[16]This is because the associated transition probabilities are time independent, so that a long time is available for performing a measurement and "finding" the particle in one of the energy eigenstates.