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Markov-Tree Model of Intrinsic Transport in Hamiltonian Systems

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A particle in a chaotic region of phase space can spend a long time near the boundary of a regular region since transport there is slow. This "stickiness" of regular regions is thought to be responsible for previous observations in numerical experiments of a long-time algebraic decay of the particle survivial probability, i.e., survival probability $\sim t^{-z}$ for large t. This paper presents a global model for transport in such systems and demonstrates the essential role of the infinite hierarchy of small islands interspersed in the chaotic region. Results for z are discussed.

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Intrinsic transport due to chaos in Hamiltonian systems is an essential element in the consideration of a number of physically important situations (e.g., confinement of plasmas with chaotic magnetic-field-line trajectories, charged-particle heating by wave electric fields, celestial mechanics, etc.). In many cases the phase space has both regular and chaotic regions. A particle initialized in a chaotic region can spend a long time in the vicinity of the boundary of a regular region since transport there is slow. This "stickiness" of the regular regions can be an essential aspect of transport in such systems.[1-6] It is our goal in this Letter to develop a global model which describes these situations.

A basic consequence of stickiness has been demonstrated in computer experiments on two-dimensional area-preserving maps done by Karney[1] and by Chirikov and Shepelyanski.[2] Imagine that there is some large region surrounded by an outermost Kolmogorov-Arnol'd-Moser (KAM) island curve[4] and that outside this island there is a connected chaotic region with interspersed smaller islands. Say that we draw a big circle which has the large island at its center and also encloses a substantial area of the outer chaotic region. Now initialize a large number N_0 of randomly chosen initial conditions in the chaotic region, and evolve them under the map. If a particle leaves our big imaginary circle, we regard it as being lost from the system. The survival probability is then taken to be $F(t)=N(t)/N_0$, where $N(t)$ is the number of particles still within the large circle at time t. From the results of Refs. 1 and 2, $F(t)$ appears to have a long-time algebraic (as opposed to exponential) decay; $F(t)\sim t^{-z}$ with $z\cong 1.5$. An essential test of a model of transport in a situation where there are chaotic and regular regions is its ability to approximate these numerically observed results. The model presented in this Letter can be viewed as an extension of a previous model[3] which correctly yielded an algebraic decay, but with a much too large decay exponent ($z=3.05$ in Ref. 3). In that model the small islands interspersed in the chaotic region were ignored. The present work shows how they can be accounted for and demonstrates that their inclusion is essential.

We describe the transport of particles in a region with regular and chaotic regions in terms of "cantori." Cantori are invariant Cantor sets that may be viewed as remnants of KAM curves that have been destroyed as a nonlinearity parameter is increased. Furthermore, certain cantori can have very small average particle flux through them. These low-flux cantori can be regarded as essentially determining the transport.[5] Numerically, what one sees[5] is that a particle, initially in a chaotic region bounded by low-flux cantori, will bounce around in that region for a long time before leaving it by crossing one of the region's bounding low-flux cantori. It then finds itself in an adjacent chaotic region bounded by other low-flux cantori, and bounces around in it for a long time before leaving, etc. Under such conditions, the particle motion is essentially a random walk from state to state, where the "states" are regions bounded by low-flux cantori.[3,5,7,8] Our model consists of a description and labeling scheme for these states, together with a specification of transition probabilities between adjacent states.

In order to describe these states, consider Fig. 1. The crosshatched regions represent areas (or islands) that are completely enclosed by an outermost KAM curve. In this schematic, for simplicity, we represent each island as a single area component, although each island is, in general, an island *chain* consisting of several area components. (An orbit, initialized in one area component of an N-component island chain, upon iteration of the map, cycles through all N components returning to its original component every Nth iterate.) Say that we fix our attention on the largest island, labeled X, in the figure. Around this island, as shown in the figure, there is an infinite sequence of low-flux cantori that encircles and accumulate on it. Between any two adjacent such cantori of this type there are possibly an infinite number of other island chains. For the moment we shall, however, assume that only one such chain is significant. (We emphasize, however, that our analysis is easily extended to an arbitrary number of such chains.) To specify this chain we locate the largest-area island chain in the area between two adjacent low-flux cantori surrounding X. We have, for example, labeled one of these chains Y in Fig. 1. The situation with regard to Y is exactly analogous to that with regard to X. In particular, there is an infinite sequence of low-flux cantori surrounding and accumulating on Y, and between each two adjacent such cantori there is again one largest island chain whose situation vis-à-vis Y can be regarded as analogous to the relationship of Y to X (although on a smaller scale), and so on *ad infinitum.*

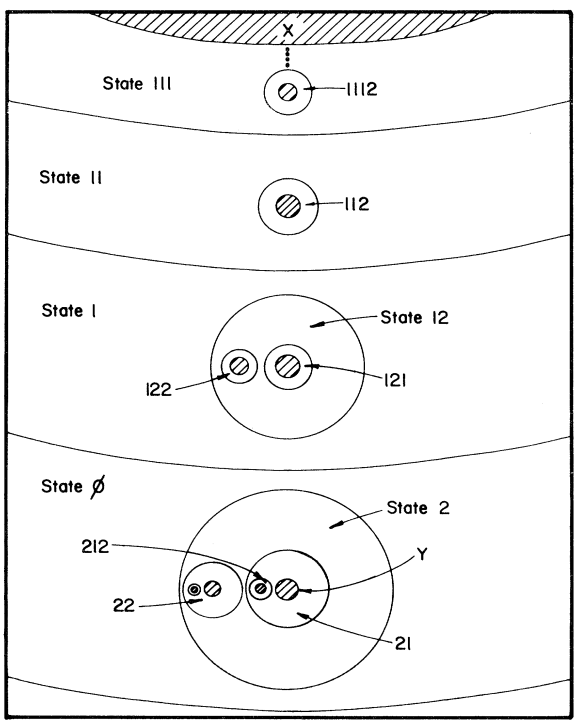

FIG. 1. Schematic illustration of connected chaotic regions of an area-preserving map bounded by low-flux cantori (states). Inaccessible regions surrounded by an outermost KAM island surface are shown crosshatched.

As already mentioned, we identify the areas bounded by low-flux cantori as "states," and we now describe a convenient method of labeling these states. First we choose some reference state which we call state ϕ (the null state). (For example, in Fig. 1 this is the outermost state encircling island X.) We then label the other states by a symbol sequence specifying their location with respect to ϕ. (States beyond the outer low-flux cantori bounding ϕ do not enter our considerations and are not labeled.) In particular, as shown in Fig. 1, we specify a state S by a sequence of a finite number of 1's and 2's $S=\sigma_1\sigma_2 \ldots \sigma_N$, where $\sigma_i=1$ or 2. $\sigma_1=1$ if the first step in making a direct path from 0 to S is to cross the inner-bounding cantorus of ϕ surrounding X; $\sigma_1=2$ if the first step is to cross the bounding cantorus of ϕ surrounding Y. If $\sigma_1=2$, then to determine σ_2, we now view Y and $S=2$ just as we viewed X and $S=0$ before. (As an illustration, we give several representative state labels in Fig. 1.) Thus, the states can be viewed as being located on a tree (or "Bethe lattice"), as shown in Fig. 2. Some additional notation will be useful for subsequent considerations. Let DS be the state symbol obtained by deletion of the last entry of S. Hence, if S has N

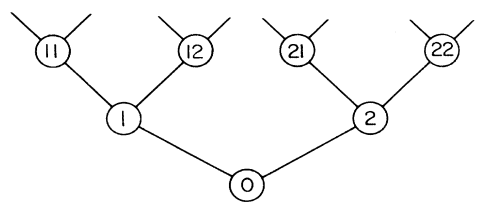

FIG. 2. Tree associated with Fig. 1.

components, then $DS=\sigma_1\sigma_2 \ldots \sigma_{N-1}$. Thus DS denotes the state just below S on the tree. Let $S1$ and $S2$ be the state symbols obtained by adding a component 1 or 2 to the end of the symbol sequence of S. Thus $S1$ and $S2$ are the two states just above S on the tree. Similarly $1S$ and $2S$ add a symbol at the beginning of the sequence. In terms of Figs. 1 and 2, we can view the numerical experiments described in Refs. 1 and 2 as initializing a large number of points in state $S=\sigma_1$ and then asking what fraction $F_{\sigma_1}(t)$ of these has never entered state ϕ after t iterates of the map. Thus, according to Refs. 1 and 2, $F_{\sigma_1}(t) \sim t^{-z}$ with $z \cong 1.5$.

To complete our model, we characterize particle motions by transition probabilities $p(S \rightarrow S')$, where $p(S \rightarrow S')$ is the "probability" that a particle in state S will be in state S' after one iterate of the map (S' is either DS, $S1$, or $S2$), and we adopt the following scaling hypotheses for $p(S \rightarrow S')$:

$$p(DS \rightarrow S)/p(S \rightarrow DS) = \begin{cases} w_1/\epsilon_1, & \text{if the last entry in } S \text{ is } 1, \\ w_2/\epsilon_2, & \text{if the last entry in } S \text{ is } 2\,, \end{cases} \tag{1}$$

$$p(S \rightarrow DS)/p(DS \rightarrow DDS) = \begin{cases} \epsilon_1, & \text{if the last entry in } S \text{ is } 1, \\ \epsilon_2, & \text{if the last entry in } S \text{ is } 2, \end{cases} \tag{2}$$

where w_1, w_2, ϵ_1, and ϵ_2 are scaling constants. Equation (2) implies that

$$p(S \rightarrow DS) = P_0 \epsilon_1^{\lambda(S)} \epsilon_2^{\rho(S)}, \tag{3}$$

where P_0 is a constant and $\lambda(S)$ and $\rho(S)$ are the number of 1's and 2's, respectively, in the symbol sequence of S. For typical (nonnoble) outermost KAM curves, the basis for the scaling hypotheses (1)–(3) is numerical.[7–9] [Actually, there are fluctuations in the ratios (1) and (2) with S, which our model ignores. Thus (1)–(3) are only approximate.[8]] Our estimated (average) values for the scaling constants are[8] $\epsilon_1 = 0.382$, $\epsilon_2 = 0.143$, $w_1 = 0.0532$, and $w_2 = 0.0142$. Figure 2 and Eqs. (1)–(3) constitute our basic model. To test the reasonableness of this model, we now apply it to the situation of the numerical experiments in Refs. 1 and 2. The method of solution is similar to that in Ref. 3.

It is convenient to use a quantity slightly different from F_{σ_1}. Namely, let $R(t;S \rightarrow S')$ be the probability that a particle in state S at time $t=0$ first reaches S' at time t. For small transition probabilities we may approximate t as a continuous, rather than as a discrete, time variable. In that case, we have $R(t;\sigma_1 \rightarrow \phi) = dF_{\sigma_1}/dt$, and $R(t;\sigma_1 \rightarrow \phi) \sim t^{-(z+1)}$. Thus by finding the long-time behavior of $R(t;\sigma_1 \rightarrow \phi)$ we can determine z. It is also convenient to introduce another quantity, $R^d(t;S \rightarrow S')$, which is the probability that a particle in state S at time zero first reaches state S' at time t without having been in any other state between time zero and time t. Following Hanson, Cary, and Meiss[3] we call $R(t;S \rightarrow S')$ the first-passage time distribution, and we call $R^d(t;S \rightarrow S')$ the *direct*-first-passage time distribution. $R^d(t;S \rightarrow S') \cong p(S \rightarrow S')\exp[-p(S)t]$, where $p(S)$ is the probability that a particle leaves S on one iterate, $p(S) \equiv p(S \rightarrow DS) + p(S \rightarrow S1) + p(S \rightarrow S2)$. For $p(S) \ll 1$, the probability that the particle has never left S is approximately $\exp[-p(S)t]$, and the above-given expression for R^d follows.

The first-passage time distribution $R(t;1 \rightarrow \phi)$ obeys

$$R(t;1 \rightarrow \phi) = R^d(t; \rightarrow \phi) + \int_0^t dt' \{R^d(t';1 \rightarrow 12) R(t-t';12 \rightarrow \phi) + R^d(t';1 \rightarrow 11) R(t-t';11 \rightarrow \phi)\}, \tag{4}$$

where we henceforth regard t as continuous. Equation (4) follows from the fact that particles arriving in $S=\phi$ can do so either directly [the term $R^d(t;1 \rightarrow \phi)$] or else by first making an "up transition" from 1 to $1\sigma_2$. The up-transition case results in the integral contribution to (4) over the time t' at which the first up transition out of 1 is made. Equation (4) can also be written as

$$R(t;1 \rightarrow \phi) = R^d(t;1 \rightarrow \phi) + [R^d(t;1 \rightarrow 12) * R(t;12 \rightarrow \phi) + R^d(t;1 \rightarrow 11) * R(t;11 \rightarrow \phi)],$$

where we use * to signify the operation of convolution in time. Similarly

$$R(t;12 \rightarrow \phi) = R(t;12 \rightarrow 1) * R(t;1 \rightarrow \phi),$$

and

$$R(t;11 \rightarrow \phi) = R(t;11 \rightarrow 1) * R(t;1 \rightarrow \phi).$$

The solution of this infinite hierarchy of coupled integral equations is simplified by use of the self-similarity relations (1)–(3). From (3) all the transition probabilities governing $R(t;11 \rightarrow 1)$ are the same as those governing $R(t;1 \rightarrow \phi)$ if the latter are multiplied by ϵ_1. Thus time is stretched by the factor ϵ_1^{-1}, or $R(t;11 \rightarrow 1) = \epsilon_1 R(\epsilon_1 t;1 \rightarrow \phi)$. Similarly, $R(t;12 \rightarrow 1) = \epsilon_1 R(\epsilon_1 t;2 \rightarrow \phi)$. Noting from (1)–(3) that $p(1S \rightarrow 1S')/p(2S \rightarrow 2S') = \epsilon_1/\epsilon_2$, we have $R(t;2$

$\rightarrow \phi) = \epsilon_2 \epsilon_1^{-1} R(\epsilon_2 \epsilon_1^{-1} t; 1 \rightarrow \phi)$. Hence, $R(t; 12 \rightarrow 1) = \epsilon_2 R(\epsilon_2 t; 1 \rightarrow \phi)$. For the sake of making the symmetry $1 \leftrightarrow 2$ manifest, we introduce $\tau = \alpha P_0 \epsilon_1 t$, $\alpha = 1 + w_1 + w_2$, and a function $h(\tau)$ given by $R(t; 1 \rightarrow \phi) \equiv P_0 \epsilon_1 h(\tau)$, which also implies $R(t; 2 \rightarrow \phi) = P_0 \epsilon_2 h(\tau)$. Inserting the above relations in (4), we obtain the fundamental integral equation of the tree model,

$$h = e^{-\tau} + e^{-\tau} * k * h, \tag{5}$$

where $k(\tau) = \alpha^{-2}[w_1 \epsilon_1 h(\epsilon_1 \tau) + w_2 \epsilon_2 h(\epsilon_2 \tau)]$. Equation (5) is a nonlinear integral equation (nonlinear because k involves h).

Noting the convolution structure of (5), we take the Laplace transform; (5) then yields

$$H(\sigma)\left[\sigma + 1 - \alpha^{-2}\left(w_1 H\frac{\sigma}{\epsilon_1} + w_2 H\frac{\sigma}{\epsilon_2}\right)\right] = 1, \tag{6}$$

where H is the Laplace transform of h, and σ is the Laplace transform variable. The long-time behavior of $h(\tau)$ is determined by the singularity of $H(\sigma)$ with the smallest real part. In particular, a power-law dependence of h on τ is reflected by a branch point of $H(\sigma)$ at $\sigma = 0$,

$$H(\sigma) = f(\sigma) + \sigma^z g(\sigma), \tag{7}$$

where f is analytic at $\sigma = 0$ and $g(0) \neq 0$. Equation (7) implies that $h(\tau) \sim \tau^{-(z+1)}$ for large τ. Substituting (7) into (6) and expanding for small σ shows that (7) is consistent with (6) with z given as the solution of

$$w_1 \epsilon_1^{-z} + w_2 \epsilon_2^{-z} = 1. \tag{8}$$

The solution of Eq. (8) for z yields $z \cong 1.96$, which is somewhat larger than the observed value. This deviation may be due to the fact that Eqs. (1) and (2) are approximate (the actual ratios fluctuate,[8,9] as previously mentioned), or it might be due to the presence of more than one island family between adjacent low-flux cantori. In the latter case, our analysis is easily generalizable, and instead of Eq. (8) we obtain[8]

$$\sum_{j=1}^{M} w_j \epsilon_j^z = 1. \tag{9}$$

The result of including more island families, (i.e., increasing M) is clearly to decrease[10] z, since each term in (9) is positive and $\epsilon_j < 1$.

In conclusion, the considerations presented here result in a primitive model for transport which indicates that the basic structure of transport is that of a random walk on a tree. Questions concerning fluctuations in the scaling parameters and the number of relevant tree branches await further study.

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[2]B. V. Chirikov and D. L. Shepelyanski, Physica (Amsterdam) **13D**, 394 (1984).

[3]J. D. Hanson, J. Cary, and J. D. Meiss, J. Stat. Phys. **39**, 327 (1985).

[4]A KAM curve is a smooth curve which is the closure of a quasiperiodic orbit of the map. For a two-dimensional map orbits cannot cross from one side of a KAM curve to the other. Thus orbits inside (outside) an outermost KAM island curve remain there forever. The case where this outermost KAM island curve has the golden-mean rotation number corresponds to Ref. 2. In general, however, the rotation number of an outermost KAM curve is not noble (J. M. Greene, J. Stark, and R. S. Mackay, to be published). The latter corresponds to the case of Ref. 1. By a noble number we mean one whose continued-fraction expansion {i.e., $a_0 + 1/[1 + a_1/(1 + a_2)/(1 + a_3/\ldots)\ldots]$, where the a_j are integers} ends in a string of ones (i.e., $a_j = 1$ for $j \geq N$ for some N).

[5]R. S. Mackay, J. D. Meiss, and I. C. Percival, Physica (Amsterdam) **13D**, 55 (1984), and Phys. Rev. Lett. **52**, 697 (1984).

[6]D. Bensimon and L. P. Kadanoff, Physica (Amsterdam) **13D**, 82 (1984).

[7]In this Letter we outline our model and its implications for the situation considered in Refs. 1 and 2. Details concerning the numerical justification of the model [e.g., Eqs. (1) and (2)] will be given elsewhere. (J. D. Meiss and E. Ott, to be published).

[8]Meiss and Ott, Ref. 7.

[9]Green, Stark, and Mackay, Ref. 4.

[10]Since these additional islands are observed to be quite small, we believe that the solution of (9) converges for $M \rightarrow \infty$.