Time Evolution and Eigenstates of a Quantum Iterative System

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We construct the evolution operator U of the quantum Chirikov map explicitly. When $2\pi\hbar = M/N$ is a rational number, corresponding to the quantum resonance condition of Casati *et al.*, we can reduce the operator U as a direct sum of independent $N\times N$ unitary matrices. We then obtain numerically the eigenstates of the same system. We describe these eigenstates in the coherent-state representation and find that they follow closely the Kolmogorov-Arnol'd-Moser curves and other classical orbits. We discuss the long-time behavior of initially localized quantum states.

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We study the behavior of quantum eigenstates in a simple quantum system at small but finite $\hbar$, and present evidence that the quantum eigenstates follow closely the classical trajectories and, in particular, the Kolmogorov-Arnol'd-Moser (KAM) curves. These eigenstates remain constant in time, and do not diffuse in either the position or the momentum space. We can express any physical state as a linear combination of these eigenstates.

The model that we have studied is the quantum Chirikov map.[1–6] This system may be described by a periodically kicked quantum rotor with the Hamiltonian

$$H = \tfrac{1}{2}p^2 + [k/(2\pi)^2]\cos(2\pi q)\textstyle\sum_n \delta(t-n). \quad (1)$$

The variable q is an angle expressed as a fraction of 2π, p is a properly scaled angular momentum, and the kicking period is chosen to be 1. Note that the momentum p between the kicks is constant. We denote the coordinate at the kick, $t=n$, as q_n, and the momenta just before and after the kick as p_n and p_{n+1}; then we obtain

$$p_{n+1} = p_n + (k/2\pi)\sin(2\pi q_n), \quad (2)$$

$$q_{n+1} = q_n + p_{n+1}(\mathrm{mod}\,1), \quad (3)$$

which are the Chirikov equations.

The quantization rule at $t=n$ is

$$[p_n, q_n] = [p_{n+1}, q_n] = -i\hbar, \quad (4)$$

which is not affected by the kick. However, $[p_n, p_{n+1}] \neq 0$.

In the following, we solve our system directly from Eqs. (2)–(4). We introduce a unitary evolution operator U obeying[7]

$$p_{n+1} = U^{-1}p_n U, \quad q_{n+1} = U^{-1}q_n U, \quad (5)$$

$$U = \exp\left[-\frac{i}{2\hbar}p_n^2\right]\exp\left[-\frac{ik}{(2\pi)^2\hbar}\cos(2\pi q_n)\right]. \quad (6)$$

One can easily show that both the quantization relation (4) and the evolution operator U are independent of n.

We study the eigenstate of the evolution operator,

$$U|\psi\rangle = e^{-i\omega}|\psi\rangle, \quad (7)$$

where ω is known as the pseudoenergy. These eigenstates are stationary under the time iteration.

We find that it is more convenient to solve (7) in momentum space. Since q is an angle, the eigenstate $\langle m|$ of p has a discrete eigenvalue m with $p = 2\pi\hbar m$. In momentum space, we can express (7) as

$$\sum_{m'=-\infty}^{\infty} U_{mm'}\phi_{m'} = e^{-i\omega}\phi_m, \quad (8)$$

where ϕ_m is the momentum-space wave function, and

$$U_{mm'} = \langle m|U|m'\rangle = \exp(-i2\pi^2 m^2\hbar)(-1)^{m'-m}J_{m'-m}(z), \quad (9)$$

with

$$z = k/(2\pi)^2\hbar. \quad (10)$$

It is difficult to evaluate the eigenvalues and the eigenvectors of an $\infty\times\infty$ with Bessel functions as its matrix elements. In the present case, we can use an additional symmetry to simplify our problem into an eigenvalue problem of finite matrices.[8]

In the classical Chirikov map, the system is invariant under $p \to p+1$. This corresponds to a transformation in the momentum eigenvalue m by

$$m \to m + (2\pi\hbar)^{-1}. \quad (11)$$

In the following, we shall choose $1/2\pi\hbar$ as an even integer N.[8] (With slight modifications, we can generalize our condition to $1/2\pi\hbar = N/M$, a rational number. The condition that $2\pi\hbar$ is a rational number is the same as the resonance condition discussed in Refs. 2–5.) Then we find that

$$U_{m+N,m'+N} = U_{mm'}. \quad (12)$$

The eigenfunction ϕ_m of U is of the Bloch wave form

$(m = s + Nl)$,

$$\phi_{s+Nl} = e^{-ial}\phi_s(a), \quad 1 \le s \le N. \tag{13}$$

It is straightforward to show that the $N \times N$ matrix

$$V_{ss'}(a) = \sum_l U_{s,s'+Nl} e^{-ila} \tag{14}$$

is unitary, and that ϕ_s, $1 \le s \le N$, given in (13) is the eigenfunction of $V(a)$ with the same eigenvalue $e^{-i\omega}$.

In the present case, we can do the summation over the Bessel functions exactly, giving[9]

$$V_{ss'}(a) = \frac{1}{N}\exp[-i\pi s^2/N + i(s'-s)a/N] \sum_{j=1}^{N} \exp\left[\frac{i2\pi(s'-s)j}{N} - iz\cos\left(\frac{2\pi j + a}{N}\right)\right], \tag{15}$$

which involves only elementary functions. In fact, we can always make such a simplification if the system is periodic in both p and q.

We can decompose an arbitrary wave function ψ_m as a linear combination of $\psi_s(a)$ $(1 \le s \le N)$ associated with the parameter a via

$$\psi_{s+Nl} = \int_0^{2\pi} (da/2\pi) e^{-ial} \psi_s(a), \tag{16}$$

where

$$\psi_s(a) = \sum_l \psi_{s+Nl} e^{ial}. \tag{17}$$

Under the operation of U, we have

$$(U^n\psi)_{s+Nl} = \int_0^{2\pi} (da/2\pi) e^{-ial} [V(a)^n \psi(a)]_s. \tag{18}$$

In other words, under an iteration, the component $\psi_s(a)$ tranforms according to the $N \times N$ matrix $V(a)$. Thus we can obtain the iterative property of ψ_m, $-\infty < m < \infty$, from the iterative property of $\psi_s(a)$, $1 \le s \le N$.

To compare the quantum states with the classical states, we describe the quantum eigenstates in the coherent-state representation.[10] A coherent-state basis is a Gaussian wave packet in both p and q spaces, centered around its expectation value (p', q'). This wave packet has a minimum combined width $\Delta p = \Delta q = \sqrt{(\hbar/2)} = 1/\sqrt{(4\pi N)}$. The wave function in such a coherent-state representation is

$$\psi(p', q') = \sum_{m=-\infty}^{\infty} (2\pi^2 N)^{1/4} \exp[-(q'^2 + ip'q')\pi N] \exp\{-\pi N[(m/N) - iq' - p']^2\} \psi_m, \tag{19}$$

where ψ_m is the wave function in p space. In Fig. 1, we plot the Chirikov map of the classical system generated by a given set of initial points. In Fig. 2, we plot the coherent-state-representation particle density $|\psi|^2$ of several eigenstates as functions of the parameter q' and p'. As we can see, in the coherent-state representation, the eigenstates follow quite closely the classical orbits. Our result agrees with the conclusion of Berry *et al.*[11] that the classical limits of the (coarsely grained) eigenstate Wigner functions[12] lie on the invariant manifolds. The $|\psi|^2$ in the coherent-state representation is in fact a coarse-grained Wigner function. The process of coarse graining is necessary here. For a small control parameter k, most eigenstates follow well-defined KAM curves.[13] For large N (small $\hbar$) and finite k, the shapes of the quantum eigenstates are still described qualitatively by the classical orbits. Classically, a KAM trajectory is a curve with no width. This trajectory is replaced in the quantum case by a wall of width $1/\sqrt{(4\pi N)}$. The chaotic regions are represented in the quantum map by ripples and no longer have well-defined wall structure. For a finite $\hbar$,

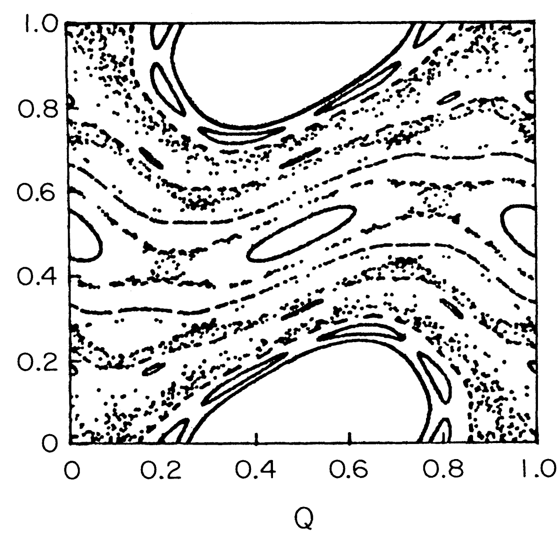

FIG. 1. Classical Chirikov map at $k = 1.0$ generated by a selected set of initial points.

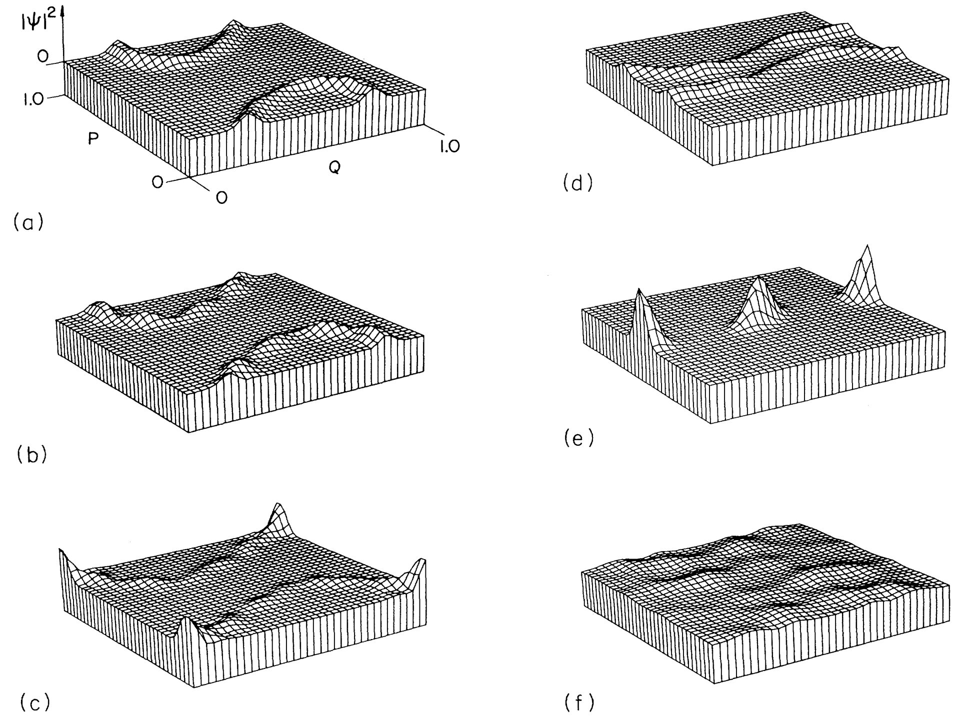

FIG. 2. The quantum eigenstates $|\psi|^2$ in the coherent-state representation, and their associated pseudoenergies ω. Each of the eigenstates is closely related to a KAM curve or other classical orbits generated by appropriate initial points. The parameters used in (a)–(e) are $k=1.0$ and $N=102(=1/2\pi\hbar)$, and in (f) are $k=10$ and $N=34$. Part (a) describes a KAM curve with $\omega=2.9921$. Part (b) describes an eight-cycle just beyond (a) with $\omega=2.1302$. Part (c) appears to represent the classical chaotic region associated with the hyperbolic fixed point at the origin with $\omega=-2.0835$. Part (d) describes the remnant of the last horizontal KAM curves associated with the golden ratio with $\omega=-2.2155$. Part (e) represents the two-cycles at $p=\frac{1}{2}$, $q=\frac{1}{2}$ and $p=\frac{1}{2}$, $q=1$ with $\omega=1.8224$. Part (f) describes a typical quantum state whose classical counterpart is chaotic with $\omega=0.2005$.

the quantum system has a resolution of $\sqrt{\hbar}$. Thus, one cannot tell for sure whether a quantum state describes a chaotic or a regular state beyond this resolution. Thus, the identification of Fig. 2(c) as a chaotic state is necessarily tentative.

We have used Eq. (18) to study the long-time behavior of states which are initially localized in both p and q. Denote the momentum eigenvalues by $2\pi\hbar(s+Nl)$, $1\leq s\leq N$, l integer. We say that the states with a given l belong to the lth cell. Expanding $\psi(a)$ in (18) in terms of the pseudoenergy $\omega_i(a)$ and the eigenfunctions $\phi_i(a)$ of $V(a)$, we have

$$(U^n)_{s+Nl}=\sum_{i=1}^{N}\int(da/2\pi)e^{-ial-in\omega_i(a)}C_i(a)[\phi_i(a)]_s, \tag{20}$$

where $C_i(a)$ are the expansion coefficients of $\psi(a)$ into the eigenfunctions $\phi_i(a)$. Note that $\omega_i(a)$ and $\phi_i(a)$ are continuous functions of a. At large n, we can show that $U^n\psi$ behaves as the superposition of N independent waves traveling with group velocities $c_i=-\omega_i'(a_i^0)$, where a_i^0 is the point of inflection of $\omega_i(a)$. Near the front of one such wave $l_i^0=\pm c_i n$, we can express (20) as

$$(U^n\psi)_{s+Nl}=\exp[ila_0-in\omega_i(a_i^0)]C_i(a_i^0)\phi_i(a_i^0)_s(\text{const}\times n)^{-1/3}\text{Ai}((l-l_i^0)/(\text{const}\times n)^{1/3}), \tag{21}$$

where $\text{Ai}(x)$ is the Airy function. The wave drops exponentially ahead of the front, and leaves an oscillatory wake behind the front, analogous to the dispersion of water waves.[14] Note that l is a measure of momentum, and n a measure of time. The relation $l_i^0=c_i n$ describes a constant acceleration in momentum, corresponding to the acceleration mode and a quadratic growth of energy.[2,3] Note that we have only studied the rational $2\pi\hbar$ (resonance case) in this paper. For irrational $2\pi\hbar$, one may expect localized eigenstates to appear in p space, as described analytically in a simple model by Berry.[15] This may explain the slower-than-classical energy growth observed in Ref. 2.

Next we ignore the l dependence and look at the diffusion in s $(1 \leqslant s \leqslant N)$ and q only in the coherent-state representation. This corresponds to identifying all the cells, and looking at its iterative behavior within this identified cell. We find that the quantum tunneling is controlled by KAM curves. A localized state in the middle of a closed KAM curve remains localized after an arbitrary number of iterations. The details of our calculations will be presented elsewhere.

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