Experimental Predictions of Lattice and Perturbative Quantum Chromodynamics

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We discuss several experimental consequences of a lattice-gauge-theory calculation of the first two moments of the π and ρ distribution amplitudes. The second moments for both mesons are considerably larger than the values obtained from QCD sum-rule techniques. We show the qualitative difference that this makes in the predicted distribution amplitude, and demonstrate the consequences for the pion form factor and the $\gamma\gamma \to \pi^0\pi^0$ cross section.

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Advances in lattice-gauge-theory calculations afford one the opportunity to combine perturbative and nonperturbative approaches to quantum chromodynamics (QCD) in order to make high-energy predictions without reference to low-energy data. We discuss here several predictions based on the first two moments of the π and ρ quark-distribution amplitudes.

QCD is generally accepted as the proper theory of the strong interactions, largely on the basis of successful predictions which are founded upon asymptotic freedom[1] and perturbative applications[2] to short-distance effects. However, these applications require nonperturbative information, often in the form of hadronic matrix elements of various strong-interaction operators, or low-energy data.

Lattice-gauge theory[3] offers the hope of a first-principles calculation of the hadronic spectrum,[4] a calculation well beyond the scope of perturbation theory. It also offers the ability to calculate the hadronic matrix elements required for perturbative analyses, which is a great step beyond treating them as unknown parameters which must be determined from data.

Three applications are discussed in this Letter, the distribution amplitude for π and ρ mesons, the pion form factor, and the cross section for $\gamma\gamma \to \pi^0\pi^0$. Before we discuss the applications, let us briefly review the distribution-amplitude formalism and the operators whose hadronic matrix elements we calculate.

Exclusive scattering processes in QCD may be expressed in terms of hard-scattering amplitudes for constituent quarks and gluons, and distribution amplitudes for the constituents of the hadronic bound states.[5] In this way, the calculation is naturally split into perturbative and nonperturbative parts, respectively. In a physical gauge the distribution amplitude $\phi(x_i,Q)$ is the probability amplitude to find constituents with momentum fractions x_i, collinear up to scale Q. For mesons it is convenient to define $\zeta = x_q - x_{\bar{q}}$. Then the moments of the distribution amplitude,

$$A^{(k)}(Q) = (2n_c)^{1/2}\int_{-1}^{1} \zeta^k \phi(\zeta,Q)\, d_\zeta, \tag{1}$$

are proportional to matrix elements of local operators:

$$\langle 0 | O^{(k)}_{\mu_0\mu_1\cdots\mu_k} | h\rangle^{(Q)} = A^{(k)}(Q)(p_{\mu_0}\cdots p_{\mu_k} - \text{traces}), \tag{2}$$

where Q is the ultraviolet cutoff in the theory, and

$$O^{(k)}_{\mu_0\mu_1\cdots\mu_k} = i^{-k}\bar{\psi}\Gamma_{\mu_0}\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_k}\psi - \text{traces}. \tag{3}$$

Here $\Gamma_\mu = \gamma_\mu\gamma_5$ and γ_μ for the π and helicity-0 ρ mesons, respectively, and D_μ is the gauge covariant derivative. The double arrow denotes the difference between the derivatives acting to the right and to the left, i.e., $\overleftrightarrow{D}_\mu = \overrightarrow{D} - \overleftarrow{D}$.

In order to calculate the matrix element of the renormalized operators in Eq. (3), we calculate bare operators on the lattice and then form the renormalized combination. This construction eliminates, at the one-loop level, the power-law divergences which occur with the lattice regulator.[6] The one-loop perturbative calculation of Ref. 6 also determines how the scale Q in the continuum is related to the lattice spacing. For the pion, we have

$$O_R^{(2)} = O_U^{(2)} - C_F\frac{\alpha}{4\pi}[1.58O^{(c)} + 4.83O^{(e)} + 2.58O^{(0)}] \tag{4}$$

with the bare operators $O^{(c)} = a^{-1}\bar{\psi}\sigma_{0j}\gamma_5\overleftrightarrow{D}^j\psi$, $O^{(e)} = a^{-1}i^{-1}\partial_0\bar{\psi}\gamma_5\psi$, and $O^{(0)} = a^{-2}\bar{\psi}i\gamma_0\gamma_5\psi$. We have also used $O^{(2)} = O^{(2)}_{000}$ as defined in Eq. (3), with subscript R (U) denoting the renormalized (unrenormalized) operator.

For the ρ, we consider somewhat different operators, but a similar expression holds:

$$O_R^{(2)} = O_U^{(2)} - C_F \frac{\alpha}{4\pi}[-13.69 O^{(c)} + 9.57 O^{(e)} + 4.28 O^{(0)}] \tag{5}$$

is free of power-law divergences to $O(\alpha)$, with $O^{(2)}$ used to denote $O^{(2)+++}$, $O^{(c)+} = a^{-1}\bar{\psi} i^{-1}\overleftrightarrow{D}^{+}\psi$, $O^{(e)+} = a^{-1}\partial_j \bar{\psi}\sigma^{+j}\psi$, and $O^{(0)+} = a^{-2}\bar{\psi} i\gamma^{+}\psi$. In all cases, tensor components denoted with a plus sign refer to light-cone coordinates, i.e., $p^+ = -ip^0 + p^i$, if the ρ is moving in the i direction. We refer the reader to Gottlieb and Kronfeld[7] for further details.

Once it is seen how the operator is renormalized, its single meson matrix element may be simply calculated on the lattice by calculation of correlation functions of the bare operators. Defining

$$C^{(n)}(t) = \left\langle \sum_{\mathbf{x}} O^{(n)}(\mathbf{x},t) O^{(0)\dagger}(\mathbf{0},0) \right\rangle, \tag{6}$$

for large t and $N_4 - t$, we have

$$C^{(n)}(t) = \langle 0|O^{(n)}|\pi\rangle \langle \pi|O^{(0)\dagger}|0\rangle (2m_\pi)^{-1}(e^{-m_\pi t} + e^{-m_\pi(N_4 - t)}), \tag{7}$$

where N_4 is the lattice periodicity in the time direction, and m_π is the pion mass. A similar expression holds for the operators which create ρ mesons. We immediately see that for long time,

$$R^{(n)}(t) = \frac{C^{(n)}(t)}{C^{(0)}(t)} = \frac{\langle 0|O^{(n)}|\pi\rangle}{\langle 0|O^{(0)}|\pi\rangle}. \tag{8}$$

Thus, it is easy to extract the ratios of matrix elements for different operators by calculation of appropriate correlation functions. For $C^{(0)}$, one fits the correlation function according to Eq. (7) in order to determine the matrix element of $O^{(0)}$.

The calculation was done with use of gauge configurations and fermion propagators which had been generated in order to calculate strong decays of mesons and baryons.[8] There were nineteen configurations on a $6^2 \times 12 \times 18$ lattice with a coupling $\beta = 5.7$. The value of the string tension was used to determine the lattice spacing, $a^{-1} = 993$ MeV. The fermionic part of the action is given by

$$S_F = \sum_x \bar{\psi}(x)\psi(x) + K \sum_{x,y} \bar{\psi}(x)[(r - \gamma_\mu)\delta_{y,x+\hat{\mu}} + (r + \gamma_\mu)\delta_{y,x-\hat{\mu}}] U(x,y)\psi(y), \tag{9}$$

where r is the chiral-symmetry–breaking parameter, and K is the hopping parameter. In this calculation, r was set to 0.5, and quark propagators were known for $K = 0.325$, 0.34, and 0.355. As usual, it was necessary to extrapolate in the hopping parameter (which determines the quark mass) to the physical value, K_p, determined by setting the pion mass equal to 140 MeV.

The numerical results for distribution-amplitude moments $A^{(0)}$ and $A^{(2)}$ are displayed in Tables I and II. Reference 7 explains how $A^{(2)}/A^{(0)}$ is calculated independently of $A^{(2)}$ and $A^{(0)}$ and, hence, why the fifth column is not the ratio of the fourth and third columns. The zeroth moments are related to the π and ρ decay constants and may be directly compared with the experimental values. This calculation had been done in previous lattice Monte Carlo simulations.[9] The first moments should vanish by charge-conjugation invariance, and on the lattice we find that $C^{(1)}$ is alternating in sign and 2 orders of magnitude smaller than $C^{(0)}$ or $C^{(2)}$. The new quantities presented in this Letter are the values for $A_\pi^{(2)}$ ($Q = 7.5$ GeV) $= 235 \pm 25$ MeV and $A_\rho^{(2)}$ ($Q = 6.8$ GeV) $= 260$

TABLE I. Results for the π extrapolated to the physical value of K. All quantities are dimensionless; physical units can be restored by use of $a^{-1} = 993$ MeV.

K	m_π^2	$A_\pi^{(0)}$	$A_\pi^{(2)}$	$\frac{A_\pi^{(2)}}{A_\pi^{(0)}}$
0.325	1.000(40)	0.243(8)	0.133(5)	0.547(20)
0.340	0.723(34)	0.215(13)	0.159(8)	0.735(40)
0.355	0.476(55)	0.184(20)	0.192(20)	1.05(10)
0.379(5)	yields K_p	0.140(28)	0.235(25)	1.37(20)
Expt.	0.020	0.133		

TABLE II. Results for the ρ extrapolated to the physical value of K. All quantities are dimensionless; physical units can be restored by use of $a^{-1} = 993$ MeV.

K	m_ρ	$A_\rho^{(0)}$	$A_\rho^{(2)}$	$\frac{A_\rho^{(2)}}{A_\rho^{(0)}}$
0.325	1.07(4)	0.382(32)	0.146(14)	0.383(30)
0.340	0.94(5)	0.346(44)	0.176(20)	0.507(30)
0.355	0.81(4)	0.312(60)	0.229(30)	0.73(11)
0.379	0.60(9)	0.254(110)	0.261(50)	0.933(20)
Expt.	0.775	0.216		

± 50 MeV. Let us explore the implications of these values.

The fact that $A_\pi^{(2)} > A_\pi^{(0)}$ implies that the distribution amplitude ϕ must be negative somewhere. For the ρ, however, $A_\rho^{(2)} \approx A_\rho^{(0)}$ and a node in the amplitude is only strongly suggested. In Fig. 1 we show the pion distribution amplitude as reconstructed from the first two moments. We also show the large-Q^2 asymptotic limit of the amplitude and the amplitude as determined in the work of Chernyak and Zhitnitsky,[10] where QCD sum rules[11] have been used to evaluate the moments. Our calculation gives a much larger value for $A^{(2)}$ than the work based on QCD sum rules. With our large value of $A^{(2)}$ there is quite a significant probability for the quark and antiquark in the meson to share their momentum equally (ϕ is big near $\zeta=0$); however, that is not the case for the QCD sum-rule result.

The pion form factor at large Q^2 is determined by the Gegenbauer moments of $\phi_\pi(\zeta,Q)$. Defining

$$\tilde{A}_\pi^{(n)}(Q)=(2n_c)^{1/2}\int_{-1}^{1} d\zeta\, C_n^{(3/2)}(\zeta)\phi(\zeta,Q), \quad (10)$$

we have,[12]

$$Q^2F_\pi(Q^2)=C_F\frac{8\pi}{n_c}\alpha_s(Q^2)\left[\sum_{n\ \mathrm{even}}\frac{2n+3}{(2+n)(1+n)}\tilde{A}_\pi^{(n)}(Q^2)\right]^2. \quad (11)$$

In Fig. 2, we show the prediction for the pion form factor as determined from our two moments and Eq. (11), but multiplying the form factor by $(1+m_\rho^2/Q^2)^{-1}$ to match the behavior of vector-dominance models[12] at low Q^2. Again we display the result as obtained by QCD sum rules as well as the asymptotic curve obtained from $A^{(0)}$. It is clear that our value of $A^{(2)}$ requires some substantial negative higher moment. The calculation of operators with yet higher derivatives seemed impractical on the size lattice that we used. Future simulations with weaker coupling, and hence smaller lattice spacing, might find such a calculation practical and interesting. In fact, if only the first two moments can be calculated, these values would provide a check of scaling. A study of scaling is particularly important because of the relatively strong coupling used here. Although $\beta=5.7$ was once thought to mark the onset of the asymptotic scaling region, a recent finite-temperature study of QCD[13] indicates that $\beta=6.15$ does. The calculation done here should be repeated by any group doing spectral calculations in that coupling region. Even a qualitative evaluation of the fourth moment would be of interest in light of the cancellation needed to bring about agreement with the pion form factor. It is also worth noting that the techniques developed here were applied to fermion propagators for gauge field configurations distributed according to the quenched approximation. As soon as this approximation can be removed from simulations, this calculation can easily be repeated with a more realistic set of gauge configurations.

The final prediction that we wish to discuss is that of pion pair production from two-photon collisions.[14] The cross section for neutral pions depends sensitively upon the shape of the pion distribution amplitude.

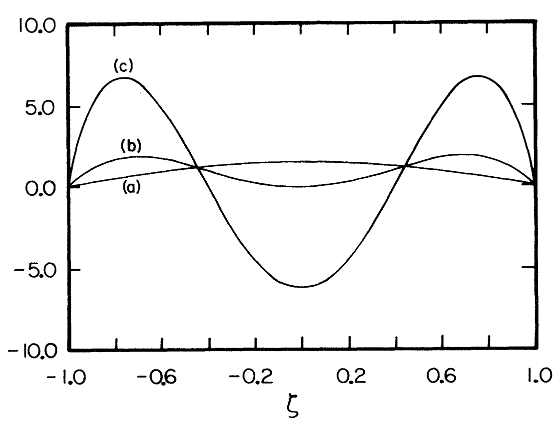

FIG. 1. The π distribution amplitude for various values of $\tilde{A}_R^{(2)}(Q)$. Curve a, the asymptotic result $\tilde{A}_R^{(2)}(Q)=0$. Curve b, the result from QCD sum rules (Ref. 10). Curve c, the result of the present numerical work.

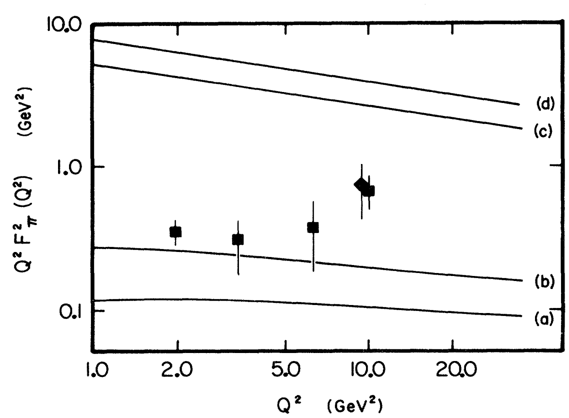

FIG. 2. The pion form factor. Curve a, asymptotic theoretical prediction. Curve b, QCD sum-rule prediction. Curves c and d, the prediction of the present work, assuming that higher moments are negligible (they are not), and using extrapolations of columns five or four of Table I, respectively. The points are from $ep\rightarrow en\pi^+$ (squares) [C. Bebek *et al.*, Phys. Rev. D **17**, 1963 (1978)] and $\psi\rightarrow\pi^+\pi^-$ (diamond) [C. G. Wohl *et al.* (Particle Data Group), Rev. Mod. Phys. **56**, No. 2, Part II (1984), tabulate $\Gamma(\psi\rightarrow\pi^+\pi^-)$ and $\Gamma(\psi\rightarrow e^+e^-)$. A trivial calculation yields the form factor].

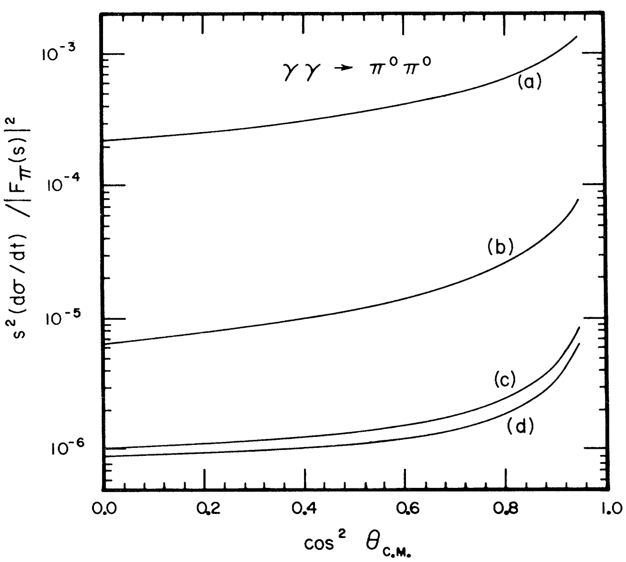

FIG. 3. Cross section for $\gamma\gamma \to \pi^0\pi^0$, scaled by the form factor as suggested in Ref. 12. Curves a–d correspond to the methods listed in Fig. 2.

Unfortunately, there is no simple formula in terms of the moments of the amplitude. It is necessary to first reconstruct the amplitude as best one can from the known moments. In Fig. 3, we display the cross section using our values for $\tilde{A}^{(0)}$ and $\tilde{A}^{(2)}$ assuming that higher moments vanish.

Lattice-gauge theory enables one to calculate matrix elements for interesting operators with covariant derivatives. Knowledge of these matrix elements is crucial input to many types of perturbative predictions. Future calculations can and should be done to check scaling and to probe higher moments than those studied here.

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