Morphology of a Class of Kinetic Growth Models

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We study a class of local probabilistic growth processes that includes the kinetic-growth algorithm for generating percolation clusters. The shapes of the growing clusters are controlled by p, the probability of growth. For $p > p_c$, the shapes are scale invariant with time and show interesting morphological features including both smoothly curved sections and straight facets. The facets are shown to be related to the problem of directed percolation and disappear below the directed-percolation threshold. A simple random-walk model for computing the shapes of our clusters is described.

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Among the many models of kinetic growth processes are a set in which the growing object develops according to some strictly local rules which govern the addition of particles to the cluster. Such models may be either deterministic, so that the distribution of new growth is completely fixed once the geometry and history of the existing structure is specified, or probabilistic, in which the growth rules are implemented according to some set of probabilities $\{P\}$. Both types of models have appeared in the study of a variety of topics including cellular automata, epidemic models, cluster growth models, etc.[1] One such model is the algorithm of Broadbent, Hammersley, Leath, and Alexandrowicz which kinetically generates percolation clusters.[2] Previous studies of the probabilistic models have generally concentrated on values of $\{P\}$ near a critical value such that the growth is marginal, often giving rise to fractallike structures.

In this Letter we wish to broaden the scope of these studies and begin the investigation of a class of probabilistic local growth models for a range of values of $\{P\}$. In particular we are interested in studying the shape of the growing cluster as a function of $\{P\}$. When $\{P\}$ is away from the critical value, such that the growth probability is enhanced, we find that the cluster grows with a very regular scale-invariant shape which reflects, in a specific way, the anisotropy of the underlying lattice, and which varies as a function of $\{P\}$. Moreover, far from the critical value of $\{P\}$, the shape varies little from one model to the next, although other features such as the density of the cluster may vary more. As we shall explain below, we have been able to relate characteristic changes in the morphology as a function of $\{P\}$ to the critical behavior of certain other growth processes such as directed percolation. Furthermore, we have been able to understand qualitatively the shape of our clusters in terms of a simple calculable model of random walks which approximates our growth processes.

In the class of models that we have studied the growth algorithm at a given time step is applied in the region of particles which were added to the cluster in the previous time step. Thus growth sites stay active for one (or, more generally, a finite number) of time steps. Such growth processes we call generational. One example of such a model (and the one that we shall focus on in most of this paper) is the following process: Place a seed particle at a site on a two-dimensional square lattice. In the first time step check the four neighbors of the seed and occupy each one, independently, with a probability p. Call the new particles the second generation. In the next time step sample the nearest neighbors of the second generation and fill these sites independently with a probability p. (A site which is a nearest neighbor of more than one growth site still has only a probability p to be filled in a given time step). Call the new particles the third generation. In the next time step, sample the nearest neighbors of the third generation, filling them with a probability p, etc. If a site is tested but not occupied in the nth time step, it may be sampled later and occupied at a later generation. The rules for this process can easily be modified to produce either site- or bond-percolation clusters. To produce site-percolation clusters, we add the condition that sites which are not filled are blocked and cannot be filled at a later time. To produce bond-percolation clusters we modify our rules so that if a site is a neighbor of more than one growth site, it is filled with a probability which depends, in an easily calculable way, on the number of neighboring growth sites. It should also be pointed out that this model is not the same as the Eden model[3] in which all perimeter sites stay active indefinitely. (If we modify our model so that all perimeter sites are active growth sites for all time, then we approach the Eden model in the limit $p \to 0$.) In Fig. 1 we show the results of computer simulations for this growth process carried out to 250 generations for a range of values of

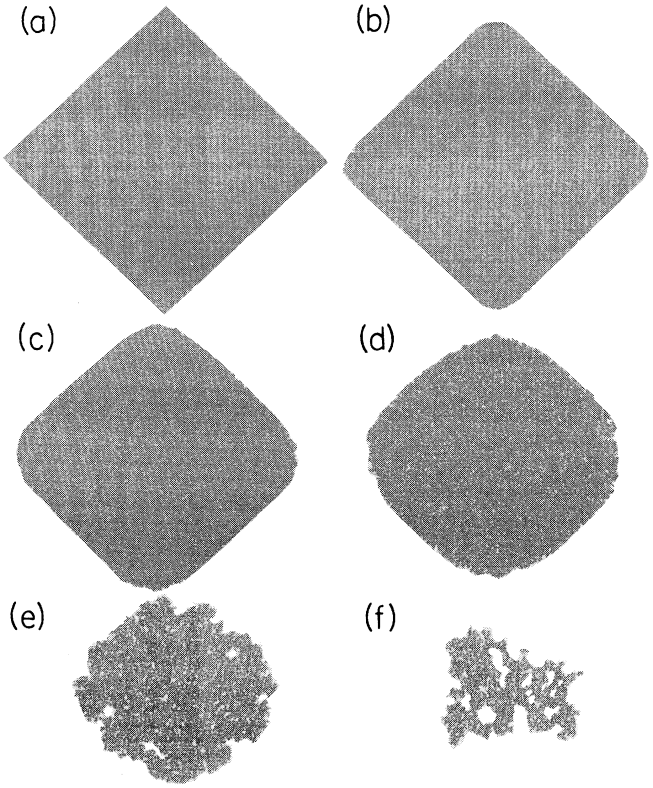

FIG. 1. Clusters grown for 250 generations according to the algorithm presented in this paper (see text for fuller explanation). (a) $p=1$, (b) $p=0.9$, (c) $p=0.8$, (d) $p=0.7$, (e) $p=0.6$, and (f) $p=0.545$. Note that with a fixed number of generations the overall cluster size decreases with decreasing p.

p. Notice that for $p \gtrsim 0.7$ the shape consists of rounded corners with intervening straight facets. The facets correspond to those points which are the maximum possible number of lattice steps (i.e., N) from the origin of the cluster in a cluster grown for N generations. The facets may also be defined as encompassing that set of sites N steps away from the origin which have a finite probability of occupation as $N \to \infty$. As we shall report in more detail elsewhere,[4] the facets are smooth in a technical sense, having statistical fluctuations of order 1, independent of N. For $p=1$ the rounded corners disappear and we have a perfect diamond. At a value of $p=p_f \cong 0.705$ the straight facets disappear and the object becomes (roughly) smoothly curved, although still showing lattice anisotropies. Fluctuations due to the finite value of p roughen the surface, but an average over many clusters will produce an object with a smooth unfaceted boundary. As we lower p below p_f, the boundary of the object becomes rougher (although still on average smooth) and the density decreases. For $p<p_c \cong 0.54$, the bulk percolation threshold for our process, the object does not grow arbitrarily large, growing only to some (average) finite size. A rough determination of p_c was made by growing, for each p, 100 clusters starting at the center of a 600×600 lattice and finding that value of p below which no cluster reached the edge. A more precise determination of p_c can be obtained by finding that value of p for which $n(s)$, the number of clusters with s particles in an ensemble of simulations, has the power-law behavior $n(s) \sim s^{1-\tau}$.[4] At $p=p_c$ the growth is marginal and fractallike, with, we believe, a Hausdorff dimension of less than 2. For all values of $p \geqslant p_c$, the growth is scale invariant. Objects grown with a value of $p > p_c$ show no tendency to change shape as the number of generations increases.[4]

These results are typical of those obtained for a variety of related growth models. Let us now try to understand the general features of this kind of growth. The two most striking qualitative features of the growth are the behavior of the facets as a function of p and the existence of marginal growth of the cluster at $p=p_c$. Let us discuss first the existence of apparently well-defined facets for $p > p_f$. Notice that for a cluster grown up to N generations these facets are defined by the equation $|l|+|m|=N$, where (l,m) is the coordinate of a site on the lattice which is the greatest distance to which the crystal can grow after N steps. We can study the facets in a little more detail, by rotating our lattice by 45° and concentrating on a single quadrant. The leading-edge growth which determines the facets now appears as a probabilistic cellular automaton advancing up a triangular lattice. We start our growth with a seed on the bottom layer of the triangular lattice, and occupy the sites on the next layer according to the rule that a site is occupied with probability p if either or both of the neighboring sites one layer below is occupied. Otherwise the site is unoccupied. We see then that the clusters so generated are just connected directed-percolation clusters which start from a point.[5] For $p > p_f$ there is apparently a finite probability that such a directed-percolation cluster will grow arbitrarily large, i.e., will make it up to the Nth level for arbitrarily large N. $p=p_f \simeq 0.705$ may therefore be identified with the threshold for directed percolation[6] at which directed percolation is marginal and the directed-percolation clusters are fractal. Thus we see that the morphological bifurcation signaled by the disappearance of facets in our growth process is controlled by the critical behavior of embedded directed-percolation clusters.

For $p \geqslant p_f$ the facet can be characterized by θ, the angle it subtends at the origin. Indeed, as $p \to p_f$, this angle approaches zero and behaves like $\theta \sim (p-p_f)^{\rho}$, where ρ is a critical exponent. In a simple random-walk model for the growth of our cluster, to be described below, $\rho=\frac{1}{2}$.[7] A more detailed analysis of the facet problem will be presented elsewhere.[4]

The second striking feature of the growth of the cluster is the absence of arbitrarily large clusters for

$p < p_c$ and the marginal, fractal nature of the cluster at $p = p_c$. The existence of marginal growth at some value of p is a common feature of many probabilistic growth models. In the specific model of Fig. 1, $p_c \cong 0.54$, and at that value of p the clusters appear to be fractal with a Hausdorff dimension $D < 2$.

It is interesting to compare the behavior of the model shown in Fig. 1 with another related model. As we stated earlier, we may easily modify the model of Fig. 1 to obtain growth algorithms for site or bond percolation. Let us compare the results of the ordinary site-percolation growth algorithm of Broadbent, Hammersley, Leath, and Alexandrowicz described above with the results of the model of Fig. 1. (Qualitatively similar results are obtained if we study the version of bond percolation described earlier.) As a function of p, this ordinary site-percolation growth model has morphological features quite similar to the model of Fig. 1. In particular, for large p there are facets which vanish at some value of p, and at a still lower value of p the growth becomes marginal and fractal. For example, a cluster grown by use of the ordinary percolation growth algorithm with $p = 0.8$ has a shape virtually identical to that of the cluster of Fig. 1(c), although the cluster in Fig. 1(c) does have a higher overall density.[4] Indeed, it is easy to demonstrate that the facets disappear at the same value of $p = p_f \cong 0.705$ in either model. The ordinary percolation growth model also becomes marginal, but this occurs at a value of $p = 0.59$, different from the value of p_c for the model of Fig. 1. It is not clear whether the Hausdorff dimension of the model of Fig. 1 at $p \cong 0.54$ is the same as the Hausdorff dimension of ordinary percolation at $p \cong 0.59$. We have, however, calculated τ for the model of Fig. 1 and find[4] a value for τ that is very close to the result for ordinary percolation[8] of $\tau \cong 0.05$.

Whether or not the marginal-growth clusters in these models turn out to be in the same universality class, it is clear that the general morphology as a function of p is the same. These general features are evidently quite robust and will appear in a wide range of related growth models. In order to understand better the origin of these results, we now introduce a simple calculable model which may be thought of as a free-field approximation to our growth models, and which reproduces these general morphological features.

If we consider one of our generational growth processes, for example, that of Fig. 1, we see that there is an approximate correspondence between the cluster pattern developed after N generations and a weighted collection of all random walks on the lattice of N steps. In particular, consider a random walk of a maximum number of N steps that begins at the origin. Each time a step is taken the walk is allowed to continue with a probability p, and terminate with a probability $1-p$. The collection of the trails of all such independent random walks is an approximation to the cluster grown in N generations. There are several ways in which this random-walk model differs from the real growth process. Among the most important are the following related facts: (1) The nature of one of the "walks" in the growth process is different from a normal random walk since it involves aspects of self-avoidance, as well as avoidance of other walks proceeding simultaneously, and (2) the walks in the growth process are not all independent. Walks which coincide along some length of their paths do not occur with independent probabilities. Despite these important differences the random-walk model gives a good qualitative description of the shape of our growth processes, as we shall now show.

Consider a specific random walk that begins at the origin of our square lattice and terminates in exactly N steps on a site with coordinates (l,m). Since the walk has a probability p to continue at each step, the probability that it does not fulfill its destiny of N steps is given by $1-p^N$. Now consider all the possible random walks of exactly N steps that begin at the origin and end at site (l,m). What is the probability that at least one such walk survives for N steps and terminates on its target site (l,m)? Since the walks are all independent, it is easy to see that this probability is just given by

$$\mathcal{P} = 1-(1-p^N)^{R(l,m;N)}, \tag{1}$$

where $R(l,m;N)$ is the number of walks that begin at the origin and terminate at (l,m) in exactly N steps and is given by

$$R(l,m;N) = \frac{(N!)^2}{[\frac{1}{2}(N+l+m)]![\frac{1}{2}(N+l-m)]![\frac{1}{2}(N-l+m)]![\frac{1}{2}(N-l-m)]!}. \tag{2}$$

Since we are interested in large N, l, and m, we can use Stirling's formula in (2). Approximating $\ln(1-p^N) \cong -p^N$, we have after a little algebra

$$\mathcal{P} = 1-\exp(-\exp\{N[\ln p + f(x,y)]\}), \tag{3}$$

where $x = l/N$, $y = m/N$, and

$$f(x,y) = \frac{1+x+y}{2}\ln\left(\frac{1+x+y}{2}\right) + \frac{1+x-y}{2}\ln\left(\frac{1+x-y}{2}\right) + \frac{1-x+y}{2}\ln\left(\frac{1-x+y}{2}\right) + \frac{1-x-y}{2}\ln\left(\frac{1-x-y}{2}\right). \tag{4}$$

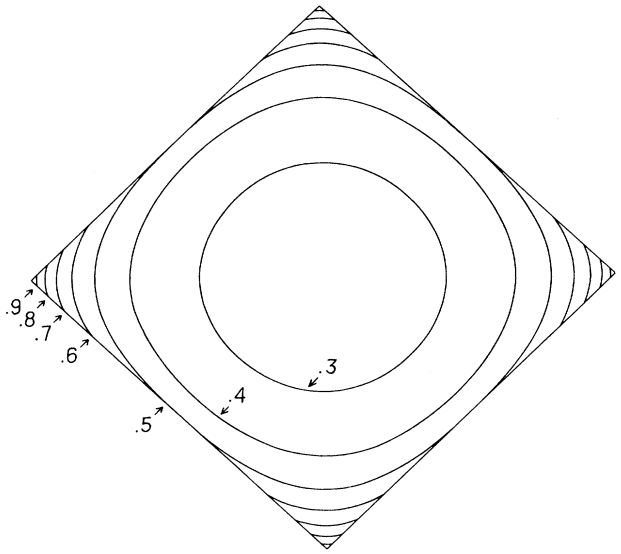

FIG. 2. Curves describing the shapes of clusters according to the free-field theory of expression (5) in the text for various values of p.

Now, physically, we must restrict ourselves to the region $|x|+|y|\leq 1$ since after N steps $|l|+|m|\leq N$. Let us fix p and study the exponent in (3). If $\ln p+f(x,y)<0$, then as $N\rightarrow\infty$, $P\rightarrow 0$, and the probability for that site to be occupied is zero. If $\ln p+f(x,y)>0$, then as $N\rightarrow\infty$, $P\rightarrow 1$, and we are assured that the cluster will grow onto the site (Nx,Ny). Therefore, the expressions

$$\ln p+f(x,y)=0, \tag{5a}$$

$$|x|+|y|\leq 1, \tag{5b}$$

determine the shape of the boundary of the growing cluster. Plots of the boundary determined by Eqs. (5) are shown in Fig. 2. For $p=1$ we have the sharp cornered diamond, consistent with our growth models. As p drops from 1 the typical shape is that of rounded corners with intervening straight facets. At $p=\frac{1}{2}$ the facets disappear. The value of p should be compared to $p=p_f=0.705$ in the growth model of Fig. 1. For $\frac{1}{4}<p<\frac{1}{2}$, the cluster grows with a smooth, unfaceted (but anisotropic) scale-invariant shape. For $p<\frac{1}{4}$ the cluster grows only to a finite size. This value of p at which the growth is "marginal" should be compared with $p=p_c\cong 0.54$ in the model of Fig. 1 (or $p\cong 0.59$ in the ordinary site-percolation model). Thus, the qualitative features of our cluster growth models are reproduced by this simple random-walk model.

The growth processes that we have studied here are very natural candidates for describing a range of kinetic physical and biological phenomena, and their morphology is very rich and intriguing. There are many very interesting questions raised by our study. One of the most important is the question of how universal the qualitative shapes of our clusters are. Given the anisotropy of the underlying lattice, what class of growth processes will give rise to clusters with a morphology similar to the ones described here? The fact that our simple random-walk model generates clusters with such shapes suggests that the general morphological behavior that we have observed here will transcend many of the details of specific-generation growth processes and will occur in a very large class of systems.

Other important questions include the connection of the models studied here with other kinds of growth models, as well as a more complete mathematical theory describing the growth of our clusters. These issues will be addressed in more detail elsewhere.[4]

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[1]See, for example, M. Eden, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probabilities, 1960,* edited by J. Neyman (Univ. of California Press, Berkeley, 1961); N. T. Bailey, *The Mathematical Theory of Infectious Diseases* (Griffin, London, 1975); S. Wolfram, Rev. Mod. Phys. **55**, 601 (1983); J. L. Cardy and P. Grassberger, J. Phys. A **18**, L267 (1985); G. MacKay and N. Jan, J. Phys. A **17**, L757 (1984); S. Havlin and R. Nossal, J. Phys. A **17**, L427 (1984); H. E. Stanley, P. J. Reynolds, S. Redner, and F. Family, in *Real-Space Renormalization,* edited by T. W. Burkhardt and J. M. J. van Leeuwen (Springer-Verlag, Heidelberg, 1982), pp. 171–208.

[2]The idea of growing percolation clusters by the sequential addition of particles has been often rediscovered. See, for example, S. R. Broadbent and J. M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 629 (1957); P. L. Leath, Phys. Rev. B **14**, 5046 (1976); Z. Alexandrowicz, Phys. Lett. **80A**, 284 (1980).

[3]See Eden, Ref. 1.

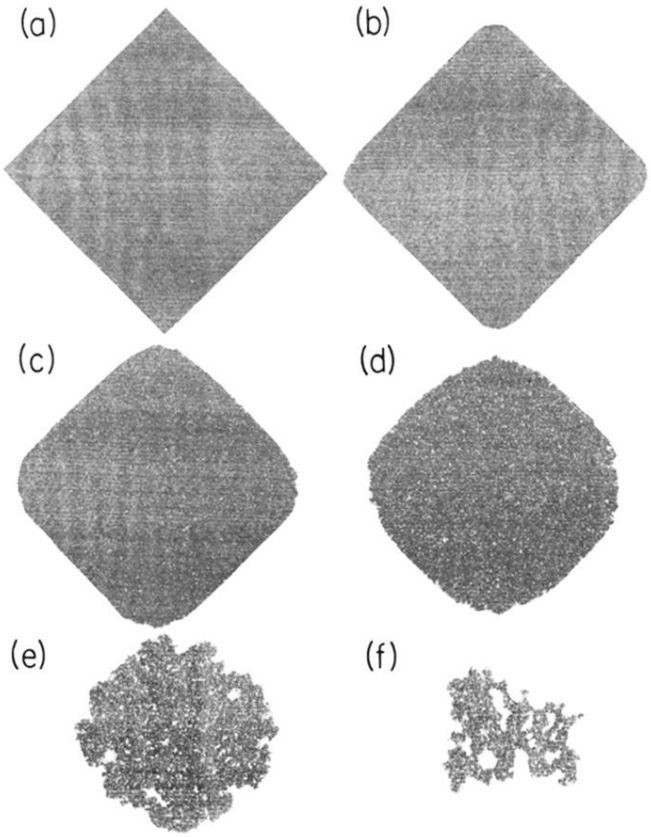
[4]R. Ziff, *et al.,* to be published.

[5]See, for example, W. Kinzel in *Percolation Structure and Processes,* edited by G. Deucher, R. Zallen, and J. Adler, Annals of the Israel Physical Society, Vol. 5 (Adam Hilger, Bristol, 1983), and references therein.

[6]J. K. Williams and N. D. Mackenzie, J. Phys. A **17**, 3343 (1984).

[7]This result follows from the random-walk growth model, to be described below. A similar result has also been derived in the context of directed percolation. See, for example, Ref. 5, and references therein.

[8]D. Stauffer, Phys. Rep. **54**, 1 (1979).



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