Exact Inequality for Random Systems: Application to Random Fields

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An inequality relating averages of generalized correlations to averages of generalized susceptibilities for Gaussian field distributions is presented. This inequality is applied to random-field systems to prove under the assumption of a continuous transition the (tree level) decoupling of the quenched two-point function. By assumption of only a power-law divergence, a lower bound for η is obtained. It rules out the possibility that some recent experimental and numerical results reflect equilibrium properties near a continuous transition.

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The problem of the lower critical dimension of the random-field Ising system, which has been the subject of many experimental and theoretical debates, seems to have been settled in favor of 2 being the lower critical dimension,[1-18] proved rigorously for $T=0$.[19] The critical behavior, however, is still controversial.

Theoretically, the critical behavior is dealt with in terms of a dimensionality reduction. Namely, the critical behavior of the random-field system is described in terms of the critical behavior of the pure system in a reduced effective dimension. Aharony, Imry, and Ma,[2] who first introduced the concept, found a reduction by 2. This result, which was later established by Young[3] and by Parisi and Sourlas,[4] suggested that the lower critical dimension is 3. The perturbative approach leading to the dimensionality reduction by 2 is to be doubted because of the possibility of Griffiths singularities at many values of the strength of the random field.[20]

Consequently one of us (M.S.) introduced an equivalent annealed system to mimic the quenched one.[16,17] The result of that approach was that the critical behavior is still given in terms of a dimensional reduction but the effective dimension d' is given by $d'=d-2+\eta(d')$ where d is the physical dimension and η the anomalous dimension of the pure system. The lower critical dimension predicted by this new relation is 2 and not 3 as predicted by the $d'=d-2$ result.

According to this result, the specific heat in three dimensions does not diverge at the transition temperature. While this prediction is not inconsistent with some experimental results that seem to allow a nondivergent specific heat,[13] an analysis of other experiments yields $\nu=1$, $\eta=\frac{1}{4}$, and a logarithmically diverging specific heat[14,15] which is compatible with a dimensional reduction from 3 to 2. The same results were obtained numerically by Young and Nauenberg.[21]

An interesting explanation was recently suggested by Shapir.[18] By use of the ideas of scaling and response he finds two hyperscaling relations. The first relation is obtained by taking into account the long-time response yielding thus an equilibrium relation $(d-2+\eta)\nu=2-\alpha$ that together with the assumption of dimensional reduction yields the effective dimension $d'=d-2+\eta(d')$. The second relation is obtained by considering short-time response, thus describing a nonequilibrium situation that may be relevant to some experiments because of the long times associated with the random system. His result for the effective dimension (assuming dimensional reduction) is $d'=d-1/\nu(d')$ if $\alpha(d')>0$ and $d'=\frac{1}{2}[d+1/\nu(d')]$ if $\alpha(d')<0$. This nonequilibrium reduction is consistent with the results of Refs. 14 and 15.

The purpose of the present Letter is to present a general theorem applicable to a very wide class of random systems, relating generalized susceptibilities and correlation functions, and to apply it to Gaussian random-field systems. Our aims are to prove the following statements: If the transition is continuous then (a) at or near the transition the decomposition $[\langle\phi_q\phi_{-q}\rangle]_{av} \doteq [\langle\phi_q\rangle\langle\phi_{-q}\rangle]_{av}$ for a small q ($\langle A\rangle$ denotes thermal average; $[A]_{av}$, ensemble average), first suggested by Aharony, Imry, and Ma[2] on the basis of a perturbation expansion, is exact; (b) assuming dimensionality reduction in conjunction with our exact result, we obtain an inequality relating d' and d; (c) with no assumption involved, we obtain on lower bound on η.

Universality suggests that the fact that the experimental results of Refs. 14 and 15, yielding the dimensional reduction from three to two dimensions, are not consistent with both inequalities implies that those measurements cannot describe equilibrium properties associated with a second-order transition.

Consider a general random system defined by the Hamiltonian

$$\beta H = H_0 + \sum_\lambda (h_\lambda \psi_\lambda^* + h_\lambda^* \psi_\lambda), \tag{1}$$

where λ is a general index. The field h_λ is a realization of a Gaussian probability distribution

$$\mathscr{P}\{h\} = [\pi h(\lambda)]^{-1/2} \prod_\lambda \exp[-|h_\lambda|^2/h(\lambda)^2], \tag{2}$$

where $h(\lambda) > 0$.

The composite fields ψ_λ are general functionals of the elementary dynamical variables. H_0 and ψ_λ do not depend on the h's. The family of systems described by Eqs. (1) and (2) is very general and includes as special cases random-field, random-temperature, random-bond, and spin-glass systems. Each of the different systems is obtained by a proper choice of H_0 and ψ_λ. The system described by Eq. (1) may be defined in a box or on a regular lattice but also on other more exotic geometric structures like amorphous systems, random nets (in which case H_0 will have an additional h-independent random character), and so on.

Consider the thermal average of ψ_λ,

$$\langle \psi_\lambda \rangle = \frac{\mathrm{tr}\{\psi_\lambda \exp[-H_0 - \sum_\lambda (h_\lambda \psi_\lambda^* + h_\lambda^* \psi_\lambda)]\}}{\mathrm{tr} \exp[-H_0 - \sum_\lambda (h_\lambda \psi_\lambda^* + h_\lambda^* \psi_\lambda)]}. \tag{3}$$

The ensemble-average λ-dependent associated susceptibility is obtained by taking the derivative of $\langle \psi_\lambda \rangle$ with respect to h_λ:

$$[\partial \langle \psi_\lambda \rangle / \partial h_\lambda]_{\mathrm{av}} = -[\langle \psi_\lambda \psi_\lambda^* \rangle]_{\mathrm{av}} + [\langle \psi_\lambda \rangle \langle \psi_\lambda^* \rangle]_{\mathrm{av}}. \tag{4}$$

On the other hand,

$$[\partial \langle \psi_\lambda \rangle / \partial h_\lambda]_{\mathrm{av}} = \int (\partial \langle \psi_\lambda \{h\} \rangle / \partial h_\lambda) \mathscr{P}\{h\} \mathscr{D}h = -\int \langle \psi_\lambda \{h\} \rangle (\partial \mathscr{P}\{h\} / \partial h_\lambda) \mathscr{D}h. \tag{5}$$

The special form of the Gaussian distribution yields

$$\partial \langle \psi_\lambda \rangle / \partial h_\lambda = h^{-2}(\lambda) \int \langle \psi_\lambda \rangle h_\lambda^* \mathscr{P}\{h\} \mathscr{D}h, \tag{6}$$

or

$$[|\langle \psi_\lambda - \langle \psi_\lambda \rangle|^2 \rangle]_{\mathrm{av}} = [\langle \psi_\lambda \psi_\lambda^* \rangle]_{\mathrm{av}} - [\langle \psi_\lambda \rangle \langle \psi_\lambda^* \rangle]_{\mathrm{av}} = -h^{-2}(\lambda) [\langle \psi_\lambda \rangle h_\lambda^*]_{\mathrm{av}}. \tag{7}$$

An obvious result is that

$$[\langle \psi_\lambda \rangle h_\lambda^*]_{\mathrm{av}} \leq 0, \tag{8}$$

which implies that the external field h_λ^* affects the composite field ψ_λ to be antiparallel to it on the average, thus lowering the energy of the system.

Now $[f^*\{h\} g\{h\}]_{\mathrm{av}}$ may be viewed as a scalar product $f \cdot g$ and, consequently, the Schwartz inequality can be applied to it:

$$|[\langle \psi_\lambda \rangle h_\lambda^*]_{\mathrm{av}}| \leq \{[\langle \psi_\lambda \psi_\lambda^* \rangle]_{\mathrm{av}} [h_\lambda h_\lambda^*]_{\mathrm{av}}\}^{1/2}. \tag{9}$$

This yields an inequality relating the λ-dependent susceptibility to the full correlation function,

$$[\langle \psi_\lambda \psi_\lambda^* \rangle]_{\mathrm{av}} - [\langle \psi_\lambda \rangle \langle \psi_\lambda^* \rangle]_{\mathrm{av}} \leq h^{-1}(\lambda) \{[\langle \psi_\lambda \rangle \langle \psi_\lambda^* \rangle]_{\mathrm{av}}\}^{1/2}. \tag{10}$$

To illustrate the usefulness of the general inequality we obtain, we apply it now to the random-field case. In this case ψ_λ is the Fourier transform of the lth spin component, ϕ_q^l, the field correlations are short ranged, and the lattice is periodic. We obtain

$$[\langle \phi_q^l \phi_{-q}^l \rangle]_{\mathrm{av}} - [\langle \phi_q^l \rangle \langle \phi_{-q}^l \rangle]_{\mathrm{av}} \leq |h|^{-1} \{[\langle \phi_q^l \rangle \langle \phi_{-q}^l \rangle]_{\mathrm{av}}\}^{1/2}. \tag{11}$$

The above inequality holds for any finite lattice and so it holds in the volume limit.

At a continuous transition $[\langle \phi_q^l \phi_{-q}^l \rangle]_{\mathrm{av}}$ and $[\langle \phi_q^l \rangle \langle \phi_{-q}^l \rangle]_{\mathrm{av}}$ diverge for small q. Consequently in that region inequality (11) leads to the decomposition

$$[\langle \phi_q^l \phi_{-q}^l \rangle]_{\mathrm{av}} = [\langle \phi_q^l \rangle \langle \phi_{-q}^l \rangle]_{\mathrm{av}} + C(q), \tag{12}$$

where $C(q)$ is negligible compared to $[\langle \phi_q^l \phi_{-q}^l \rangle]_{\mathrm{av}}$ as $q \to 0$. If $[\langle \phi_q^l \phi_{-q}^l \rangle]_{\mathrm{av}}$ and $[\langle \phi_q^l \rangle \langle \phi_{-q}^l \rangle]_{\mathrm{av}}$ are described for small q by power-law divergences like $q^{-\Gamma_1}$ and $q^{-\Gamma_2}$, respectively, then $\Gamma_1 = \Gamma_2 = \Gamma$. This decomposition was suggested by Aharony, Imry, and Ma[2] on the basis of a perturbation expansion, which involves ignoring all diagrams which diverge more weakly than the leading order. The number of the neglected diagrams is infinite and so they can, in principle, accumulate to contribute to the leading order. Furthermore, even the full diagrammatic expansion is doubtful.[20] Equation (12) implies that the corrections are at most of order $q^{-\Gamma/2}$, so that the decomposition is exact and does not actually depend on any kind of expansion.

Inequality (11) suggests immediately an exponent inequality. If the correlation function diverges at the transition as $q^{-\Gamma}$ and the q-dependent susceptibility as $q^{-2+\eta}$, we obtain

$$2 - \eta \leq \Gamma/2. \tag{13}$$

Let us now assume dimensional reduction. As suggested by the work of Aharony, Imry, and Ma[2] and Parisi and Sourlas[4] and as may be easily checked, the dimensional reduction manifests itself in the behavior of the susceptibility in q space and the full correlation in real space. Namely,

$$[\langle \phi_q^l \phi_{-q}^l \rangle]_{\mathrm{av}} - [\langle \phi_q^l \rangle \langle \phi_{-q}^l \rangle]_{\mathrm{av}} \sim q^{-2+\eta(d')}, \tag{14}$$

and

$$[\langle \phi^l(r) \phi^l(0) \rangle]_{\mathrm{av}} \sim r^{-[d'-2+\eta(d')]}. \tag{15}$$

By definition of Γ,

$$\Gamma = d - d' + 2 - \eta(d'). \tag{16}$$

With the use of (13),

$$4-2\eta(d') \leqslant d-d'+2-\eta(d'), \tag{17}$$

or

$$d' \leqslant d-2+\eta(d'). \tag{18}$$

It was suggested before,[16,17] on the basis of an assumption of the existence of a Gaussian region and scaling, that relation (11) should be an order-of-magnitude equality leading to an equality in Eq. (18), thus giving the result $d'=d-2+\eta(d')$, that was obtained by a different method, as the only consistent dimensional reduction. Here, however, our only assumption is dimensional reduction and obviously the result $d'=d-2$ cannot be ruled out.

Inequality (13) can be used to obtain an exact lower bound on η regardless of any assumption. Since $\langle \phi_i^2 \rangle = 1$ we conclude that $\Gamma < d$ (otherwise $[\langle \phi_i^2 \rangle]_{av} = \infty$), obtaining

$$\eta \geqslant (4-d)/2. \tag{19}$$

($\Gamma < d$ may also be obtained from the obvious but unproven fact that the average correlation function is decreasing in distance.)

It is very easy to check that neither (18) nor the slightly less restrictive (because it does not depend on any dimensional-reduction assumption) (19) is consistent with the results of Refs. 14 and 15 and the numerical calculations.[21] The connection between the Gaussian field distribution we treat and the actual experimental unknown short-range field distribution is provided by universality so that the critical behavior expected in both cases is the same. Since our results are exact, one explanation that comes to our mind is that while we consider static properties, these experiments[14,15] and numerical results[21] are probably affected by the special slow dynamics of the system. This explanation is consistent with the arguments presented by Shapir.[18]

Another explanation suggested by Young and Nauenberg[21] is that the transition is actually first order. During the preparation of this Letter we have learned that the above possibility was first raised on the basis of some experimental results by Birgeneau *et al.*[22]

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