Derivative Expansion of the Effective Action

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(Received 15 October 1985)

A method is described for the calculation of derivative terms in the effective action to one loop. Sample calculations are made of the two- and four-derivative terms for a single scalar field, and of the two-derivative terms for O(N) scalar field theory.

PACS numbers: 11.10.Ef

In order to study the behavior of quantum field theories in which states with nonconstant-classical-field configurations may be significant, it is important to understand the quantum corrections to the classical action. Typical problems include solitons in QCD and in the Higgs sector of the standard model[1-4] and other problems in which derivative interactions play an important role.[3,5] Previous methods used to calculate the effective gradient terms have entailed either reconstruction of the effective action from some carefully chosen amplitudes,[2] or complicated functional methods.[6] Two recent series of papers,[3,4] however, have presented a much simpler systematic procedure for calculating derivative terms in the effective action to one-loop approximation. These two approaches are different, but closely related. (However, they give different results for the example of a pure scalar field theory.) The aim of the present paper is to present a hybrid method, which I believe is algebraically simpler than either of the two.

The starting point for the calculation is the functional determinant for the one-loop effective action,[7]

$$\mathcal{S}_1=\int d^4x\mathcal{L}_1=\frac{i}{2}\ln\det\mathcal{D}=\frac{i}{2}\mathrm{Tr}\ln\mathcal{D}, \tag{1}$$

where

$$\mathcal{D}_{ij}=\frac{\delta^2\mathcal{S}}{\delta\phi_i\delta\phi_j}(\phi)$$

is the inverse propagator in the presence of a classical background field ϕ. The trace in (1) is straightforward to evaluate if $\phi(x)$ is a constant field, $\phi(x)=\phi(0)$; one obtains the Coleman-Weinberg result for the effective potential.[8] If $\phi(x)$ is not constant, however, we must contend with the fact that the terms appearing in $\mathcal{D}$ do not commute with each other.

As a simple example, consider the scalar field Lagrangean

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2-\frac{1}{3!}\mu\phi^3-\frac{1}{4!}\lambda\phi^4. \tag{2}$$

In this case we have

$$\mathcal{D}(x-y)=[-\partial^2-U(x)]\delta(x-y), \tag{3}$$

where $U(x)=m^2+\mu\phi+\frac{1}{2}\lambda\phi^2$. Following the approach of Ref. 3, we write $U(x)=U(0)+\delta U(x)$, so that the spatial dependence is contained in $\delta U(x)$, with $\delta U(0)=0$. To simplify the calculation, it is convenient to use the identity

$$\mathrm{Tr}\ln(A+B)=\mathrm{Tr}\ln A+\int_0^1 dz\,\mathrm{Tr}\frac{1}{A+zB}B. \tag{4}$$

Putting $A=\mathcal{D}_0\equiv[-\partial^2-U(0)]\delta(x-y)$ and $B=-\delta U(x)\delta(x-y)$, we have

$$\begin{aligned}\mathcal{S}_1&=\frac{i}{2}\mathrm{Tr}\ln\mathcal{D}_0-\frac{i}{2}\int_0^1 dz\,\mathrm{Tr}\left[\frac{1}{\mathcal{D}_0-z\delta U(x)}\delta U(x)\right],\\&=\bar{\mathcal{S}}_1+\tilde{\mathcal{S}}_1.\end{aligned} \tag{5}$$

$\bar{\mathcal{S}}_1=\int d^4x\bar{\mathcal{L}}_1$ is the one-loop contribution to the effective potential for constant field $\phi(0)$. $\tilde{\mathcal{S}}_1$ includes the effects of nonconstant-field configurations.

In order to perform the trace in $\tilde{\mathcal{S}}_1$, we must disentangle $\mathcal{D}_0$ and $\delta U(x)$, which do not commute. Following Ref. 4, a convenient way to do this is to make the following substitutions in the denominator of (5):

$$\delta U(x)\to\delta U(x+i\,\partial/\partial p),\quad -\partial^2\to p^2.$$

We then expand the integrand of $\tilde{\mathcal{S}}_1$ in a power series in z and $\partial/\partial p$; in this manner we obtain for the gradient part of the effective Lagrangean

$$\tilde{\mathcal{L}}_1=-\frac{i}{2}\int_0^1 dz\int\frac{d^4p}{(2\pi)^4}\sum_{n=0}^{\infty}z^n\left[G\sum_{m=1}^{\infty}\frac{i^m}{m!}\partial_{\mu_1}\cdots\partial_{\mu_m}\delta U(x)\frac{\partial^m}{\partial p_{\mu_1}\cdots\partial p_{\mu_m}}\right]^n G\delta U(x), \tag{6}$$

where

$$G=\frac{1}{p^2-U(0)-z\delta U(x)}$$

and

$$\tilde{\mathscr{S}}_1=\int d^4x\tilde{\mathscr{L}}_1.$$

A few comments about this expression are in order:

(1) p^2 here is just a numerical quantity—not an operator—and so p^2 and $\delta U(x)$ commute.

(2) The momentum derivatives act on everything to their right, while the spatial derivatives act only on the adjacent $\delta U(x)$.

(3) The $n=0$ term can be absorbed into $\bar{\mathscr{L}}_1$ (the effective potential term), which then becomes

$$\bar{\mathscr{L}}_1=\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\ln[p^2-U(x)],$$

the effective potential for constant field $\phi(x)$ [instead of $\phi(0)$].

(4) The Taylor expansion of $\delta U(x)$ starts at $m=1$, the $m=0$ term having been absorbed into G. This is the reason for introducing the z integral rather than just expanding the logarithm in (1) directly—we eliminate a number of terms at intermediate stages of the calculation. They reappear later, but then the combinatoric factors are found simply as coefficients in a Taylor series in z. It may also be noted that $iG(z)$ interpolates between the field-dependent propagator for background field $\phi(0)$ at $z=0$ and that for $\phi(x)$ at $z=1$.

(5) Finally, (6) is an expansion for $\tilde{\mathscr{L}}_1$ at x. We may anticipate a general form for $\tilde{\mathscr{L}}_1$ as

$$\tilde{\mathscr{L}}_1=a_1\frac{[\partial_\mu U(x)]^2}{U(0)}+a_2\frac{[\partial^2U(x)]^2}{U(0)^2}+a_3\frac{[\partial^2U(x)][\partial_\mu U(x)]^2}{U(0)^3}+a_4\frac{[\partial_\mu U(x)]^4}{U(0)^4}+\dots, \tag{7}$$

where the a_i are themselves given as a series in $\delta U(x)/U(0)$,

$$a_i=\sum_{n=0}^{\infty}b_{i,n}\left(\frac{\delta U(x)}{U(0)}\right)^n. \tag{8}$$

$\delta U(x)/U(0)$ is not necessarily a small quantity. However, the effective action is translation invariant, and so we may just as well evaluate $\tilde{\mathscr{L}}_1$ at $x=0$ as anywhere else. When we do so, all the terms in (8) with $n>0$ vanish, since $\delta U(0)=0$. The point $x=0$ is arbitrary, and so we may rewrite (7) as

$$\tilde{\mathscr{L}}_1=b_1\frac{[\partial_\mu U(x)]^2}{U(x)}+b_2\frac{[\partial^2U(x)]^2}{U(x)^2}+b_3\frac{[\partial^2U(x)][\partial_\mu U(x)]^2}{U(x)^3}+b_4\frac{[\partial_\mu U(x)]^4}{U(x)^4}+\dots, \tag{9}$$

where b_i here corresponds to $b_{i,0}$ in (8). The upshot of all of this is that, having brought the terms in the expansion (6) into the canonical form (7), we may then discard all terms containing undifferentiated factors of $\delta U(x)$, and replace $U(0)$ by $U(x)$ [of course, $\partial_\mu\delta U(x)=\partial_\mu U(x)$, also].

To illustrate the simplicity of (6), we will compute $b_1,\dots,b_4$ in (9). For b_1, we need to keep only terms involving two powers of $\delta U(x)$; expanding (6) we find a single contributing term,

$$-\frac{i}{2}\int_0^1dz\int\frac{d^4p}{(2\pi)^4}z\frac{i^2}{2!}G\frac{\partial^2}{\partial p_\mu\partial p_\nu}G\delta U(x)\partial_\mu\partial_\nu\delta U(x). \tag{10}$$

We may neglect all but the leading behavior in z in the integrand, since higher powers of z bring in extra factors of $\delta U(x)$. The momentum integral is straightforward; a few handy formulas are

$$\frac{\partial}{\partial p_\mu}G=-2p^\mu G^2,\quad \frac{\partial^2}{\partial p_\mu\partial p_\nu}G=-2G^2(g_{\mu\nu}-4p_\mu p_\nu G),$$
$$\int\frac{d^4p}{(2\pi)^4}p^{2s}G^n=(-1)^{n-s}i[U(0)+z\delta U(x)]^{-(n-s-2)}\frac{\Gamma(s+2)\Gamma(n-s-2)}{16\pi^2\Gamma(n)}. \tag{11}$$

We thus obtain for the two-derivative term

$$-\frac{1}{24}\frac{1}{16\pi^2}\frac{\delta U(x)\partial^2\delta U(x)}{U(0)}\left[1+O\left(\frac{\delta U(x)}{U(0)}\right)\right] \tag{12}$$

or, adding a total derivative, we find

$$b_1=\frac{1}{24}\frac{1}{16\pi^2}. \tag{13}$$

Adding this term to the classical Lagrangean, we obtain

$$\frac{1}{2}Z(\phi)(\partial_\mu\phi)^2=\frac{1}{2}\left[1+\frac{1}{12}\frac{1}{16\pi^2}\frac{(\mu+\lambda\phi)^2}{m^2+\mu\phi+\frac{1}{2}\lambda\phi^2}\right](\partial_\mu\phi)^2, \tag{14}$$

for the one-loop–corrected kinetic-energy term, in agreement with Fraser,[3] Chan,[4a] and Iliopoulos, Itzykson, and Martin.[6]

The computation of the four-derivative terms, b_2, b_3, and b_4, is only slightly more involved. In this case, we need to keep track of up to four powers of $\delta U(x)$, and expanding (6) to the appropriate order, we have

$$\begin{aligned}\tilde{L}_1=&(\partial^2\text{ term})-\frac{i}{2}\int_0^1 dz\int\frac{d^4p}{(2\pi)^4}G\delta U(x)\left\{\frac{z}{4!}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\delta U(x)\frac{\partial^4}{\partial p_\mu\,\partial p_\nu\,\partial p_\rho\,\partial p_\sigma}G\right.\\
&+\frac{z^2}{3!}\partial_\mu\delta U(x)\partial_\nu\partial_\rho\partial_\sigma\delta U(x)\left[\frac{\partial}{\partial p_\mu}G\frac{\partial^3}{\partial p_\nu\,\partial p_\rho\,\partial p_\sigma}G+\frac{\partial^3}{\partial p_\nu\,\partial p_\rho\,\partial p_\sigma}G\frac{\partial}{\partial p_\mu}G\right]\\
&+\frac{z^2}{(2!)^2}\partial_\mu\partial_\nu\delta U(x)\partial_\rho\partial_\sigma\delta U(x)\frac{\partial^2}{\partial p_\mu\,\partial p_\nu}G\frac{\partial^2}{\partial p_\rho\,\partial p_\sigma}G+\frac{z^3}{2!}\partial_\mu\delta U(x)\partial_\nu\delta U(x)\partial_\rho\partial_\sigma\delta U(x)\\
&\times\left.\left[\frac{\partial}{\partial p_\mu}G\frac{\partial}{\partial p_\nu}G\frac{\partial^2}{\partial p_\rho\,\partial p_\sigma}G+\frac{\partial^2}{\partial p_\rho\,\partial p_\sigma}G\frac{\partial}{\partial p_\mu}G\frac{\partial}{\partial p_\nu}G+\frac{\partial}{\partial p_\nu}G\frac{\partial^2}{\partial p_\rho\,\partial p_\sigma}G\frac{\partial}{\partial p_\mu}G\right]\right\}\\
&+\text{higher derivatives.}\end{aligned} \tag{15}$$

The momentum integrals are evaluated by use of the formulas (11). We can then expand the integrand in powers of z, retaining up to 4 factors of $\delta U(x)$, and do the z integral. The result is

$$\begin{aligned}\tilde{\mathscr{L}}_1=&(\partial^2\text{ term})+\frac{1}{2}\frac{1}{16\pi^2}\delta U(x)\left\{\frac{1}{60}\frac{\partial^4\delta U(x)}{U(0)^2}\left[\frac{1}{2}-\frac{2}{3}\frac{\delta U(x)}{U(0)}+\frac{3}{4}\left(\frac{\delta U(x)}{U(0)}\right)^2\right]\right.\\
&-\frac{1}{15}\frac{\partial_\mu\delta U(x)\partial^\mu\partial^2\delta U(x)}{U(0)^3}\left(\frac{1}{3}-\frac{3}{4}\frac{\delta U(x)}{U(0)}\right)-\frac{1}{45}\frac{[\partial_\mu\partial_\nu\delta U(x)]^2}{U(0)^3}\left(\frac{1}{3}-\frac{3}{4}\frac{\delta U(x)}{U(0)}\right)\\
&\left.+\frac{1}{40}\frac{\partial_\mu\delta U(x)\partial_\nu\delta U(x)\partial^\mu\partial^\nu\delta U(x)}{U(0)^4}\right\}+\text{higher derivatives.}\end{aligned} \tag{16}$$

By adding total derivatives and discarding terms of the form (9) with extra factors of $\delta U(x)$, we finally obtain

$$\tilde{\mathscr{L}}_1=\frac{1}{16\pi^2}\left\{\frac{1}{24}\frac{(\partial_\mu U)^2}{U}+\frac{1}{240}\frac{(\partial^2 U)^2}{U^2}-\frac{1}{180}\frac{\partial^2 U(\partial_\mu U)^2}{U^3}+\frac{1}{480}\frac{(\partial_\mu U)^4}{U^4}\right\}+\text{higher derivatives.} \tag{17}$$

This result may be seen to be in agreement with equations 2.30, 2.32, and 2.34 of Fraser[3] by expanding U in terms of ϕ and adding a total derivative.

This technique is easily extended to cases in which $\mathscr{D}$ carries internal or Lorentz indices. As a brief example, consider the case of $O(N)$ scalar field theory, with the Lagrangean

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m^2\phi^2-(1/4!)\lambda(\phi^2)^2, \tag{18}$$

for which we have

$$\mathscr{D}_{ij}(x-y)=\{[-\partial^2-U_1(x)]Q_{1ij}(x)+[-\partial^2-U_2(x)]Q_{2ij}(x)\}\delta(x-y), \tag{19}$$

where we define

$$U_1=m^2+\tfrac{1}{2}\lambda\phi^2,\quad U_2=m^2+\tfrac{1}{6}\lambda\phi^2,\quad Q_{1ij}=\phi_i\phi_j/\phi^2,\quad Q_{2ij}=\delta_{ij}-\phi_i\phi_j/\phi^2, \tag{20}$$

and we have

$$Q_1Q_2=0\quad\text{and}\quad Q_a^2=Q_a. \tag{21}$$

Again, we split $\mathscr{D}$ into a translation-invariant piece,

$$\mathscr{D}_0=\{[-\partial^2-U_1(0)]Q_1(0)+[-\partial^2-U_2(0)]Q_2(0)\}\delta(x-y), \tag{22}$$

and the remainder,

$$\delta\mathscr{D} = -\delta(U_1Q_1 + U_2Q_2)\delta(x-y). \tag{23}$$

The expression analogous to (6) is then

$$\tilde{\mathscr{L}}_1 = -\frac{i}{2}\hat{\mathrm{Tr}}\int_0^1 dz \int \frac{d^4p}{(2\pi)^4}\sum_{n=0}^{\infty} z^n\{[G_1Q_1(0) + G_2Q_2(0)]\overline{\delta U}\}^n[G_1Q_1(0) + G_2Q_2(0)] \times [\delta U_1(x)Q_1(0) + \delta U_2(x)Q_2(0) + \Delta U(x)\delta Q(x)], \tag{24}$$

where $\hat{\mathrm{Tr}}$ denotes the O(N) trace,

$$G_a = \frac{1}{p^2 - U_a(0) - z\delta U_a(x)},$$

$$\overline{\delta U} = \Delta U(x)\delta Q(x) + \sum_{m=1}^{\infty}\frac{i^m}{m!}\partial_{\mu_1}\cdots\partial_{\mu_m}[\delta U_1(x)Q_1(0) + \delta U_2(x)Q_2(0) + \Delta U(x)\delta Q(x)]\frac{\partial^m}{\partial p_{m_1}\cdots\partial p_{\mu_m}}, \tag{25}$$

$$\Delta U(x) = U_1(x) - U_2(x), \quad \delta Q \equiv \delta Q_1 = -\delta Q_2.$$

The identity (21) allows us to rewrite (24) as

$$\tilde{\mathscr{L}}_1 = \tilde{\mathscr{L}}_1[U_1] + (N-1)\tilde{\mathscr{L}}_1[U_2] + \text{terms involving } \delta Q, \tag{26}$$

where $\tilde{\mathscr{L}}_1[U_a]$ is expression (6) with U_a substituted for U, and the "terms involving δQ" are the remaining terms, which all contain at least one factor of δQ. We easily obtain the two-derivative terms,

$$\tilde{\mathscr{L}}_1 = \frac{1}{16\pi^2}\left\{\frac{\lambda^2}{24}(\phi_i\partial_\mu\phi^i)^2\left[\frac{1}{U_1} + \frac{N-1}{9}\frac{1}{U_2}\right] - \frac{1}{2}\left(\frac{(\partial_\mu\phi_i)^2}{\phi^2} - \frac{(\phi_i\partial_\mu\phi^i)^2}{\phi^4}\right)\left[\frac{U_1U_2}{\Delta U}\ln\frac{U_1}{U_2} - \frac{U_1+U_2}{2}\right]\right\} + \ldots. \tag{27}$$

It is amusing to note that the leading behavior for large N is just given by the expression for a single scalar field, since (26) is then dominated by $N\tilde{\mathscr{L}}_1[U_2]$.

I would like to thank Mary K. Gaillard for her interest in and comments on this work, and Zvi Bern for helpful discussions. This work was supported in part by the National Science Foundation under Grant No. PHY-84-06608 and by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

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