Induced Angular Momentum in (2+1)-Dimensional QED

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We show that in the presence of a magnetic flux tube, the ground state of Dirac fermions carries a net angular momentum. We use a trace-identity analysis to calculate it. We find the induced angular momentum, for a flux tube with one quantum of flux, to be fractionized to $\frac{1}{8}$. This is the first example of an induced external quantum number.

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(2+1)-dimensional quantum electrodynamics has come under some scrutiny in the past few years. It was observed that, in contradistinction to its (3+1)-dimensional analog, there exists an additional term which can be appended to the usual Lagrangean, that serves to give the gauge field a mass.[1] This so-called topological mass term, explicitly given by

$$\mathscr{L}=\tfrac{1}{4}m\epsilon^{\mu\nu\sigma}A_\mu\partial_\nu A_\sigma,$$

although not manifestly gauge invariant, changes only by a total divergence under a gauge transformation, and hence its contribution to the action is gauge invariant. Further it was shown that, in the presence of Dirac fermions with mass κ, the topological mass term is automatically generated in the effective action for the gauge fields, obtained by functional integration over the fermions.[2] The induced mass of the gauge bosons is

$$m=(\mathrm{sgn}\kappa)/2\pi.$$

Recently it has been proven[3] that this result is exact; there are no higher-order corrections. A consequence of the induced topological mass term is that there exists a contribution to the induced fermionic current in the ground state, for given background fields,

$$\langle j^\mu\rangle=\langle\bar\psi\gamma^\mu\psi\rangle=[(\mathrm{sgn}\kappa)/4\pi]\epsilon^{\mu\nu\sigma}\partial_\nu A_\sigma,$$

with a net induced charge

$$Q=\int d^2x\,\langle j^0\rangle=\tfrac{1}{2}\,(\mathrm{sgn}\kappa)(\Phi/2\pi).$$

Now, if we imagine adiabatically switching on a magnetic flux tube from the vacuum, the induced charge can be calculated by integration of the charge flux through the surface at infinity.[4] However, during the switching process, the time-varying magnetic field necessarily induces an azimuthal electric field. It is natural to ask whether the action of this electric field on the induced charge can induce any angular momentum.

We consider Dirac fermions interacting with a cylindrically symmetric, static, background magnetic flux tube. The fermionic Lagrangean is

$$\mathscr{L}_F=\bar\psi(x)[i\gamma^\mu(\overleftrightarrow{\partial}_\mu+IA_\mu)-\kappa]\psi(x),$$

where $\psi(x)$ is a two-component complex spinor, $A_\mu(x)$ is the U(1) gauge field, and $\overleftrightarrow{\partial}_\mu=\frac{1}{2}(\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu)$. An explicit representation of the gamma matrices is

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad\gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad\gamma^2=\begin{pmatrix}0&i\\i&0\end{pmatrix}.$$

We work with $A_0(x)=0$, $\dot A_i(x)=0$, and a rotationally symmetric gauge field. That is, for any rotation R_i^j,

$$R_i^jA_j(x_k)=A_i(R_k^lx_l).$$

Because of this property, the fermionic action is invariant under rotations. The corresponding conserved quantum number, the angular momentum, is given by

$$M=\int d^2x\,\psi^\dagger(x)\epsilon^{ij}(-ix_i\overleftrightarrow{\partial}_j+\tfrac{1}{8}i[\gamma_i,\gamma_j])\psi(x)$$
$$=\int d^2x\,\psi^\dagger(x)(L+S)\psi(x),$$

which is just the sum of the orbital and the spin angular momenta. We take the gauge field to have the form

$$A_i(x)=\epsilon_{ij}x^jf(r),$$

where $f(r)$ is a function of the radial coordinate only. When $f(r)\sim1/r^2$, the magnetic field vanishes. We assume that this is the case for $r>R$, and $f(r)$ can be an arbitrary nonsingular function for $r<R$. Since the gauge field is single valued, there is no compelling reason to drop this requirement on the single-particle wave functions.[5] Consequently the eigenvalues of the single-particle angular momentum operator are integral.

We want to calculate the expectation value, $\langle M\rangle$, in the fermionic Fock ground state for the given background. Formally, this is just the mode sum over the Dirac sea of the expectation value of the single-particle angular momentum operator in the single-particle energy levels. We renormalize this expression by vacuum subtraction, that is by subtracting at the gauge fields set equal to zero. Define

$$\mathscr{M}(A_i)=\int d^2x\sum_{E<0}\psi_E^\dagger(x)(L+S)\psi_E(x)\Big|_0^{A_i}.$$

Tentatively we want to identify $\mathscr{M}(A_i)$ with $\langle M\rangle$;

however, even though this is a once-subtracted expression, we must regulate it to render it well defined. We will use point-splitting regularization. This can destroy some of the transformation properties enjoyed by the formal expression $\langle M\rangle$. $\langle M\rangle$ is even under charge conjugation and has a specific transformation property under gauge transformations. The transformation property under gauge transformations can be preserved by multiplication of any point-split expression by

$$\exp\left[i\int_{x'}^{x''} A_\mu dx^\mu\right],$$

where x' and x'' are the two split points. For charge conjugation, we simply define $\langle M\rangle$ by projection on the charge-conjugation–even part of a $\mathcal{M}(A_i)$. Therefore we define

$$\langle M\rangle = \tfrac{1}{2}[\mathcal{M}(A_i)+\mathcal{M}(-A_i)],$$

and evaluate the right-hand side by (gauge covariant) point-splitting regularization. The order in which we perform these manipulations, projecting on the charge-conjugation–even part and regularizing by point splitting, is of course immaterial. Since

$$\psi_{-E}(x)|_{-A_i} = i\gamma^2\psi_E(x)|_{A_i},$$

we get

$$\langle M\rangle = -\tfrac{1}{2}\int d^2x \sum_{\text{all }E}(\text{sgn}E)\psi_E^\dagger(x)(L+S)\psi_E(x)|_0^{A_i},$$

which is like an angular-momentum–weighted spectral asymmetry. We will omit the vacuum subtraction from now on. Then the regulated, point-split expression that we must analyze is

$$\langle M\rangle = -\tfrac{1}{2}\int d^2x \lim_{x',x''\to x}\exp\left[i\int_{x'}^{x''} A_\mu dx^\mu\right]\sum_{\text{all }E}(\text{sgn}E)\psi_E^\dagger(x')i\epsilon^{ij}\{\tfrac{1}{2}(x_i'\overleftarrow{\partial}_j' - x_i''\overrightarrow{\partial}_j'')+\tfrac{1}{8}[\gamma_i,\gamma_j]\}\psi_E(x''),$$

which is gauge covariant and charge-conjugation even. We only split points in the spatial hypersurface. Now we express

$$\text{sgn}E = (2/\pi)\int_0^\infty d\omega\, E/(E^2+\omega^2)$$

and use completeness of the wave functions to obtain

$$\langle M\rangle = -\tfrac{1}{2}\int d^2x \lim_{x',x''\to x}\exp\left[i\int_{x'}^{x''} A_i\, dx^i\right]\frac{2}{\pi}\int_0^\infty d\omega\,\tfrac{1}{2}\,\text{tr}\langle x''|\left\{L+S,\frac{H}{H^2+\omega^2}\right\}|x'\rangle,$$

where

$$H=\gamma^0[-i\gamma^i(\partial_i+iA_i)+\kappa] = H^0+\kappa\gamma^0$$

is the Hamiltonian, and now

$$L+S = i\epsilon^{ij}(-x_i\partial_j+\tfrac{1}{8}[\gamma_i,\gamma_j])$$

is the angular momentum operator. Then using

$$\text{tr}\langle x''|\left\{L+S,\frac{H}{H^2+\omega^2}\right\}|x'\rangle = \frac{i\kappa}{\sigma}\,\text{tr}\langle x''|\left\{L+S,\gamma^0\frac{1}{H^0+i\sigma}\right\}|x'\rangle,$$

where

$$\sigma^2=\kappa^2+\omega^2,$$

we obtain

$$\langle M\rangle = -\tfrac{1}{2}\int d^2x \lim_{x',x''\to x}\exp\left[i\int_{x'}^{x''} A_i\, dx^i\right]\frac{2}{\pi}\int_0^\infty d\omega\,\frac{i\kappa}{2\sigma}\,\text{tr}\langle x''|\left\{L+S,\gamma^0\frac{1}{H^0+i\sigma}\right\}|x'\rangle.$$

Now we use a trace identity, which can be proven by straightforward generalization of similar identities used in the analysis of the axial anomaly,[6] the index theorem of Callias,[6] and fermion-number fractionization,[7] to bring the right-hand side to a tractable form:

$$\langle M\rangle = -\tfrac{1}{2}\int d^2x\, \frac{2}{\pi}\int_0^\infty d\omega \frac{i\kappa}{2\sigma} \lim_{x',x''\to x} \exp\left(i\int_{x'}^{x''} A_i\, dx^i\right)\frac{1}{2i\sigma}$$
$$\times\left\{(\partial_i''+\partial_i')\operatorname{tr}\langle x''|i\gamma^i\left\{L+S,\frac{1}{H^0+i\sigma}\right\}|x'\rangle\right.$$
$$\left.-\operatorname{tr}[\mathbf{A}(x'')-\mathbf{A}(x')]\,\langle x''|\left\{L+S,\frac{1}{H^0+i\sigma}\right\}|x'\rangle+2\operatorname{tr}\langle x''|\left\{L+S,\gamma^0\right\}|x'\rangle\right\}.$$

The remaining is straightforward. The first term becomes a total divergence in the coincidence limit, which is integrated to the surface at infinity. Since $A_i \sim 1/r$, we need only the first few terms in the perturbative expansion of the Green's function. The second term is like an anomaly term which can be evaluated exactly by use of the short-distance structure of the Green's function. The last term does not contribute since all field-independent terms are removed by the vacuum subtraction, and we use the relation $x\,\delta(x)=0$. In evaluating this expression we must remember to include the contribution from the expansion of the gauge-invariance factor, that all charge-conjugation–odd terms, that is terms odd in $A_i(x)$, vanish, and that all field-independent terms are irrelevant as they are removed by the vacuum subtraction. We must evaluate

$$\langle M\rangle = -\frac{\kappa}{4\pi}\int d^2x\int_0^\infty d\omega\,\frac{1}{\sigma^2}\lim_{\xi,\delta\to 0}\operatorname{tr}\big(i(\boldsymbol{\partial}^{x+\xi}+\boldsymbol{\partial}^{x+\delta})\{G^{(2)}+iA_i(\xi-\delta)^iG^{(1)}-\tfrac{1}{2}[A_i(\xi-\delta)^i]^2G^{(0)}\}$$
$$+(\partial_jA_i-\partial_iA_j)\gamma^j(\xi-\delta)^i[G^{(1)}+iA_i(\xi-\delta)^iG^{(0)}]\big),$$

where $G^{(i)}$ is the ith-order perturbative expansion of the Green's function

$$G(x+\xi,x+\delta)=\langle x+\xi|\{L+S,(H^0+i\sigma)^{-1}\}|x+\delta\rangle,$$

and where I have put $x'=x+\delta$ and $x''=x+\xi$. The calculations are tedious and not illuminating; they will be reserved for a lengthier, more detailed presentation. I find that the angular momentum density is finite and unambiguous, as is the total angular momentum. The expression for the angular momentum is

$$\langle M\rangle = [(\operatorname{sgn}\kappa)/32\pi]\int d^2x\,\{(\delta^{lk}\delta^{ij}+\epsilon^{kl}\epsilon^{ij}+\epsilon^{ki}\epsilon^{lj}+\delta^{ik}\delta^{lj})[\partial_i(A_jx_kA_l)+(\partial_jA_i-\partial_iA_j)x_kA_l]$$
$$+2(\delta^{ik}\delta^{jl}-\epsilon^{il}\epsilon^{kj}-\epsilon^{ij}\epsilon^{kl})(\partial_jA_i-\partial_iA_j)x_kA_l+(\delta^{ik}\delta^{jl}-\epsilon^{il}\epsilon^{kj}-\epsilon^{ij}\epsilon^{kl})\partial_i(A_jx_kA_l)\}.$$

Then replacing

$$A_i(x)=\epsilon_{ij}x^jf(r),$$

we find that the first and last terms give vanishing contribution while the second term gives

$$\langle M\rangle=\tfrac{1}{8}(\operatorname{sgn}\kappa)(\Phi/2\pi)^2,$$

where

$$\Phi=\int d^2x\,r^{-1}[r^2f(r)]'=\int d^2x\,B$$

is the total magnetic flux.

We observe that the result depends only on the total flux which is a topological invariant. The result is charge-conjugation even. This may seem counterintuitive since one might expect the angular momentum to behave like a magnetic dipole and change sign when $B\to -B$. However, I remark that this is the angular momentum of the induced charge, which is odd under charge conjugation, rendering the angular momentum even. Finally, if magnetic flux tubes are dynamically stable, this would be the angular momentum induced on the flux tube. Even if magnetic flux were quantized in units of 2π (the usual flux quantum), the total angular momentum induced by the fermions would be fractional.

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