Comment on "Renormalization, Unstable Manifolds, and the Fractal Structure of Mode Locking"

In a recent Letter Cvitanović *et al.*[1] have used renormalization-group techniques to examine the fractal dimension of quasiperiodic winding numbers of a circle map at the onset of chaos. Earlier analyses had predicted a universal scaling behavior at a *particular* irrational winding number.[2] Cvitanović *et al.* predict that universal scaling should also hold between *any* two frequency-locked intervals. In an experiment on a nonequilibrium chemical reaction in a stirred flow reactor we have found good agreement with this scaling prediction.

In the theory[1] one can start with any pair of locked intervals with winding numbers P/Q and P'/Q'. The length of the gap between them is denoted by s_T. Next the locked interval $(P+P')/(Q+Q')$ is found, and the gaps between the new interval and the preceding ones are denoted s_1 and s_2, respectively. The construction is continued until a large number of gap sizes s_i is found. The fractal dimension D is then obtained from the formula $\sum_i (s_i/s_T)^D = 1$. Surprisingly, estimates of D based on only two gaps are accurate to within 1%.[1]

In our experiment we observe, as the flow rate is varied, a sequence of periodic states, each consisting of a combination of small- and large-amplitude oscillations.[3] The ratios P/Q of the number of small amplitude oscillations to the total number of oscillations per period form a staircase as a function of flow rate. Interpreting the ratios P/Q to be winding numbers,[3] we can apply the theory of Cvitanović *et al.* to our data, as Fig. 1 illustrates. The fourteen pairs of states P/Q and P'/Q' for which the "Farey tree" state $(P+P')/(Q+Q')$ was observed with the intervals s_1 and s_2 at least 3% wide in control-parameter range yield $D = 0.87 \pm 0.02$ (weighted average and standard deviation), in good accord with the predicted value, 0.868 ± 0.002.

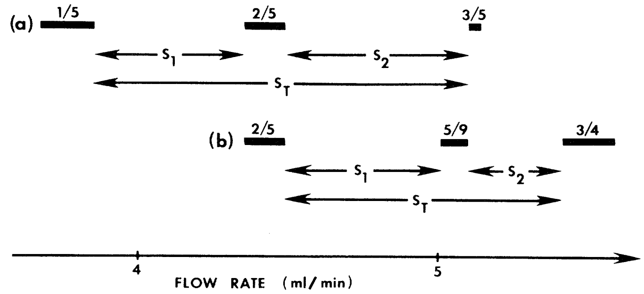

FIG. 1. Examples of the intervals used in computing the fractal dimension D from $(s_1/s_T)^D + (s_2/s_T)^D = 1$. Example (a) gives $D - 0.86 \pm 0.02$ and (b) gives 0.87 ± 0.03; the uncertainties arise from the 0.3% uncertainty in the determination of the transition points.

The agreement between theory and experiment found over a wide range in winding numbers is remarkable since the system with its thirty or more chemical species is certainly not strictly described, as in the theory, by a circle map with a cubic inflection point. For example, in a few cases we have two or more distinct successive states with the *same* ratio P/Q.[3] However, unique values of the winding numbers for our data have been obtained by Arneodo and Argoul,[4] who interpreted the data in terms of a chemical model that gives frequency locking on a torus. Using their interpretation of a winding number, we obtain $D = 0.85 \pm 0.04$. Another concern is that the theory of Cvitanović *et al.* applies in principle only at criticality of the circle map, where the frequency-locked intervals have the full measure of the winding-number values and the quasiperiodic intervals have zero measure. Although our measurements along an arbitrarily chosen path in control parameter space could not be expected to correspond exactly to criticality, the absence of hysteresis between the observed transitions and the absence of chaos or quasiperiodicity in our observations lend support to the interpretation of our measurements in terms of a torus near criticality.

A recent elegant experiment by Stavans, Heslot, and Libchaber[5] on a driven convecting fluid demonstrated the scaling behavior predicted to occur near irrational winding numbers; they carefully adjusted the winding number to be very near either the golden mean or the silver mean. The present experiment suggests that scaling may apply to a fairly wide range of winding numbers and control-parameter values in systems that exhibit frequency locking. It will be interesting to examine many other systems in which staircases of frequency locking have been observed to determine how universal the value of D is.

We thank Nathan Kreisberg for helpful discussions. This research is supported by the Department of Energy, Office of Basic Energy Sciences.

Harry L. Swinney and J. Maselko
Department of Physics
University of Texas
Austin, Texas 78712

Received 8 October 1985
PACS numbers: 05.45.+b, 05.70.Ln, 47.70.Fw, 82.20.Mj

[1]P. Cvitanović, M. H. Jensen, L. P. Kadanoff, and I. Procaccia, Phys. Rev. Lett. **55**, 343 (1985).

[2]M. J. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, Physica (Amsterdam) **5D**, 370 (1983); D. Rand, S. Ostlund, J. Sethna, and E. Siggia, Phys. Rev. Lett. **49**, 132 (1982).

[3]J. Maselko and H. L. Swinney, Physica Scr. **T9**, 35 (1985).

[4]A. Arneodo and F. Argoul, to be published.

[5]J. Stavans, F. Heslot, and A. Libchaber, Phys. Rev. Lett. **55**, 596 (1985).

